
The numbers between brackets in the margin represent the marks assigned to the question. The maximum grade is 100.

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1. Parametrize the following surfaces **and** determine the coordinates of the unit normal vector to the surface at any point.

(15) (a) $S = \{(x, y, z) \in \mathbb{R}^3;\ x^2 + z^2 = 16\}$.

(15) (b) $2x + 3y - z = 17$.

(15) (c) $(x-2)^2 + (y-3)^2 + (z-1)^2 = 16$.

(15) (d) $\dfrac{x^2}{4} + \dfrac{(y-3)^2}{4} + z^2 = 1$.

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2. In this exercise, we prove the following

Theorem 1. *Suppose $\Omega \subset \mathbb{R}^N$ is a path-connected domain and that $\mathbf{F} : \Omega \to \mathbb{R}^N$ is a continuous vector field on Ω. If $\mathbf{F}$ is a path-independent vector field, then $\mathbf{F}$ is the gradient of some scalar-valued function f. That is, there exists $f : \Omega \to \mathbb{R}$ such that $\nabla f(x) = \mathbf{F}(x)$ for all $x \in \Omega \subset \mathbb{R}^N$.*

Recall that " $\mathbf{F}$ is path-independent" means

Definition 1. $\mathbf{F}$ *is said to be path independent if for any piecewise continuous curve (C), originating at a point A and ending at a point B, the line integral $\displaystyle\int_{(C)} \mathbf{F} \cdot dr$ depends only on the points A and B. In other words, if C_1 and C_2 are two different piecewise continuous curves, having the same initia terminal points, then*

$$\int_{C_1} \mathbf{F} \cdot dr = \int_{C_2} \mathbf{F} \cdot dr.$$

It is not difficult to see that Definition 1 can be replaced by the following definition for path-independence:

Definition 2. **F** *is said to be path independent if for any* **closed** *piecewise continuous curve* C*, we have*

$$\int_C \mathbf{F} \cdot dr = 0.$$

We state the following fact (you can think about a simple proof, without writing it in your paper)

Proposition 1. *If* $\mathbf{u}_1$ *and* $\mathbf{u}_2$ *are two vectors in* $\mathbb{R}^N$ *such that* $\mathbf{u}_1 \cdot \mathbf{v} = \mathbf{u}_2 \cdot \mathbf{v}$ *for all* $\mathbf{v} \in \mathbb{R}^N$*, then*

$$\mathbf{u}_1 = \mathbf{u}_2.$$

Lastly, we recall that the gradient of a scalar function g enjoys the following

Proposition 2. *Let* $\mathbf{v}$ *be a unit vector in* $\mathbb{R}^N$ *and let* $g : \Omega \to \mathbb{R}$ *be a* C^1*-function. Then, the directional derivative of* g *in the direction of* $\mathbf{v}$ *satisfies*

$$\forall x \in \mathbb{R}^N,\ D_{\mathbf{v}} g(x) = \frac{\partial g}{\partial \mathbf{v}}(x) = \nabla g(x) \cdot \mathbf{v}.$$

In what follows, we proceed to prove Theorem 1 through some steps (that we list below as questions to be answered in this assignment). Remember, we need to find a function f so that $\nabla f = \mathbf{F}$ based on the assumption that $\mathbf{F}$ is path-independent.

Suppose $\Omega \subset \mathbb{R}^N$ is a path-connected domain and that $\mathbf{F} : \Omega \to \mathbb{R}^N$ is a **path-independent** continuous vector field on Ω. **Fix some point a** in Ω. **Let** $f : \Omega \to \mathbb{R}$ be the function defined by

$$\forall \vec{x} \in \Omega, \quad f(\vec{x}) := \int_{\gamma[\mathbf{a},\vec{x}]} \mathbf{F} \cdot dr, \tag{1}$$

where $\gamma[\mathbf{a}, \vec{x}]$ is any differentiable curve from **a** to $\vec{x}$ (recall that F is path-independent. So, the form/geometry of γ does not matter in (1). What matters is only the origin **a** and the terminal $\vec{x}$.)

(5) (a) Let $\mathbf{v}$ be a unit vector in $\mathbb{R}^N$. Explain why, for any $h \in \mathbb{R}$, we have

$$\int_{\gamma[\mathbf{a},\vec{x}+h\mathbf{v}]} \mathbf{F} \cdot dr - \int_{\gamma[\mathbf{a},\vec{x}]} \mathbf{F} \cdot dr = \int_{\gamma[\vec{x},\vec{x}+h\mathbf{v}]} \mathbf{F} \cdot dr.$$

(5) (b) We know that the direction derivative $\frac{\partial f}{\partial \mathbf{v}}(\vec{x})$ is

$$\frac{\partial f}{\partial \mathbf{v}}(\vec{x}) := \lim_{h\to 0} \frac{f(\vec{x}+h\mathbf{v}) - f(\vec{x})}{h}. \tag{2}$$

Use (2) and part (a) to show that

$$\frac{\partial f}{\partial \mathbf{v}}(\vec{x}) := \lim_{h\to 0} \frac{1}{h} \int_{\gamma[\vec{x},\vec{x}+h\mathbf{v}]} \mathbf{F} \cdot dr. \tag{3}$$

(5) (c) At this stage, we will look into the limit in (3). Here, it is worth remembering that $\gamma[\vec{x},\vec{x}+h\mathbf{v}]$ can be arbitrarily chosen (provided differentiability). To compute the limit in (3), we choose $\gamma[\vec{x},\vec{x}+h\mathbf{v}]$ to be the least complicated path from $\vec{x}$ to $\vec{x}+h\mathbf{v}$. That would be the **line segment** originating at $\vec{x}$ and terminating at $\vec{x}+h\mathbf{v}$. Using t as a parameter, show that $r(t) = \vec{x} + t\mathbf{v}$ with $0 \le t \le h$ is a suitable parametrization of the **line segment** $\gamma[\vec{x},\vec{x}+h\mathbf{v}]$.

(5) (d) Using the parametrization in (c), conclude that the limit in (3) can be written as

$$\frac{\partial f}{\partial \mathbf{v}}(\vec{x}) = \lim_{h\to 0} \frac{1}{h} \int_0^h \mathbf{F}(\vec{x}+t\mathbf{v}) \cdot \mathbf{v}\, dt. \tag{4}$$

(5) (e) We note that the integrand in the RHS of (4) can be viewed scalar function of one variable (t). Let's call, for the fixed $\vec{x}$ and $\mathbf{v}$, $\phi(t) := \mathbf{F}(\vec{x}+t\mathbf{v}) \cdot \mathbf{v}$. So (4) reads

$$\frac{\partial f}{\partial \mathbf{v}}(\vec{x}) = \lim_{h\to 0} \frac{1}{h} \int_0^h \phi(t)\, dt.$$

Let Φ be an antiderivative of ϕ. Show, using Φ, that

$$\lim_{h\to 0} \frac{1}{h} \int_0^h \phi(t)\, dt = \phi(0).$$

(5) (f) Conclude from (e) and (d) that

$$\frac{\partial f}{\partial \mathbf{v}}(\vec{x}) = \mathbf{F}(\vec{x}) \cdot \mathbf{v}. \tag{5}$$

(5) (g) Use (f) and Proposition 2 to conclude that

$$\nabla f(\vec{x}) \cdot \mathbf{v} = \mathbf{F}(\vec{x}) \cdot \mathbf{v}. \tag{6}$$

(5) (h) Recall now that **v** was arbitrary (we fixed an arbitrary unit vector **v** in part (a) above). Use Proposition 1 to conclude that $\nabla f(\vec{x}) = \mathbf{F}(x)$ and thus complete the proof of Theorem 1.

TOTAL MARKS: 100