The Riesz Representation Theorem
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1 Statement and proof of the theorem

Let H be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$, and T a bounded linear functional on H (a bounded operator from H to the field, $\mathbb{R}$ or $\mathbb{C}$ (over which H is defined).

The Riesz Representation Theorem reads:

Theorem 1. *If T is a* **bounded linear functional** *on a Hilbert space H then there exists some $g \in H$ such that for every $f \in H$ we have*

$$T(f) = \langle f, g \rangle.$$

Moreover, $\|T\| = \|g\|$ (here $\|T\|$ denotes the operator norm of T, while $\|g\|$ is the Hilbert space norm of g.)

Proof. Let's assume that H is separable for now. It's not much harder to prove Theorem 1 for any Hilbert space, but the separable case makes nice use of the ideas we developed regarding Fourier analysis. Also, let's just work over $\mathbb{R}$. Since H is separable, we can choose an orthonormal basis φ_j, $j \geq 1$, for H. Let T be a bounded linear functional and set

$$a_j := T(\varphi_j).$$

Choose $f \in H$, let $c_j = \langle f, \varphi_j \rangle$, and define $f_n = \sum_{j=1}^{n} c_j \varphi_j$. Since the φ_j form a basis we know that

$$\|f_n - f\|_H \to 0 \text{ as } n \to +\infty.$$

Since T is linear we have

$$T(f_n) = \sum_{j=1}^{n} a_j c_j. \tag{1}$$

As T is bounded, we have

$$|T(f_n) - Tf| \leq \|T\| \|f_n - f\|_H \to 0, \text{ as } n \to \infty. \tag{2}$$

Thus $\{Tf_n\}_n$ converges in R to Tf. That is,

$$T(f) = \lim_{n\to\infty} T(f_n) = \sum_{j=1}^{\infty} a_j c_j. \tag{3}$$

In fact, the sequence $\{a_j\}$ must itself be square-summable. To see this, first note that since $|T(f)| \leq \|T\|\|f\|$ we have

$$\sum_{j=1}^{\infty} a_j c_j \leq \|T\| \left(\sum_{j=1}^{\infty} c_j^2 \right)^{1/2}. \tag{4}$$

The above holds for any square-summable sequence $\{c_j\}$ (since any such $\{c_j\}$ corresponds to an element of H). So now we pick $N \in \mathbb{N}$ and take a sequence $\{c_j\}$ such that

$$c_j = a_j \text{ for } 1 \leq j \leq N, \quad c_j = 0 \text{ for } j > N.$$

Clearly this $\{c_j\}$ is square-summable and plugging it into (4) yields

$$\sum_{j=1}^{N} a_j^2 \leq \|T\| \left(\sum_{j=1}^{N} a_j^2 \right)^{1/2}. \tag{5}$$

Hence,

$$\left(\sum_{j=1}^{N} a_j^2 \right)^{1/2} \leq \|T\|. \tag{6}$$

Thus $\{a_j\}$ is square-summable, since the sequence of partial sums is bounded above.

Since $\{a_j\}$ is square-summable, it follows that $g := \sum_{j=1}^{\infty} a_j \varphi_j$ is well-defined as an element of H. From (3), we have

$$\forall f \in H, \quad T(f) = \sum_{j=1}^{\infty} a_j c_j = \sum_{j=1}^{\infty} a_j \langle f, \varphi_j \rangle = \langle f, \sum_{j=1}^{\infty} a_j \varphi_j \rangle = \langle f, g \rangle.$$

Inequality (6) (after passing to the limit $N \to \infty$) implies that $\|T\| \geq \|g\|$. On the other hand, Cauchy-Schwatz inequality implies that $|\langle f, g\rangle| \leq \|f\|\|g\|$. This means $\frac{|T(f)|}{\|f\|} \leq \|g\|$ and hence $\|T\| \leq \|g\|$. Therefore, $\|T\| = \|g\|$ and this completes the proof of Riesz theorem. □

2 An application of Riesz representation theorem in differential equations

This examples illustrates how functional analytic methods are used in PDE (though the example is for an ODE). Consider the ODE

$$-u''(x) + b(x)u(x) = f(x) \tag{7}$$

on the interval $0 < x < 1$, where $b(x) \geq \delta > 0$ for some δ; assume the functions b and f are continuous on $[0, 1]$. We want a solution to equation (7) with $u'(0) = u'(1) = 0$ (other boundary conditions are possible).

Variational/weak formulation of (7). If we multiply (7) by a C^1 function ϕ and integrate the first term, $-u''\phi$ by parts from $x = 0$ to $x = 1$, we obtain

$$\int_0^1 (u'(x)\phi'(x) + b(x)u(x)\phi(x))\ dx = \int_0^1 f(x)\phi(x)\ dx. \tag{8}$$

Equation (8) must hold for any $\phi \in C^1([0, 1])$, if u is a $C^2(0, 1)$ solution to equation (7) that is continuous on $[0, 1]$. Conversely, if for a given C^2 function u we find that (8) holds **for all** ϕ, then u must be a solution to equation (7), for if we "undo" the integration by parts in (8) we obtain

$$u(1)\phi'(1) - u(0)\phi'(0) + \phi(x)(-u''(x) + b(x)u(x)) = \phi(x)f(x) \tag{9}$$

for all ϕ. A popular PDE argument (when (9) holds **for all** test functions ϕ) then shows that $u'(0) = u'(1) = 0$ and equation (7) must hold.

We are going to show that there is a unique solution to equation (8). Such a "solution" will not necessarily be twice-differentiable as required by equation (7), but it will satisfy equation (8). Equation (8) is often called the "weak" formulation of the problem (7).

Finding a weak solution (i.e. a solution to (8)). Define the inner product

$$\langle v, w\rangle := \int_0^1 \left(v'(x)w'(x) + b(x)v(x)w(x)\right)\, dx$$

on the space $C^1[0,1]$.

Exercise. Show that this is indeed an inner product–observe the use of the assumption $b(x) \geq \delta > 0$ when showing that $\langle g, g\rangle = 0$ iff $g \equiv 0$ on $[0,1]$. Note that the presence of b in the inner product makes no much difference from the inner product with no $b(x)$.

Let H denote the completion of this space. In other words, H is such that $C^1[0,1] \subset H$ and H is complete when equipped with the norm induced by the above inner product. The space H is a Hilbert space, and can be interpreted (if need be) as a subspace of $C([0,1])$.

Define a functional $T : H \to \mathbb{R}$ by

$$T(\phi) = \int_0^1 f(x)\phi(x)\, dx.$$

You can easily check that T is bounded on H (use Cauchy-Schwarz). **From the Riesz Representation Theorem** it then follows that there must exist some function $u \in H$ such that

$$T(\phi) = \langle u, \phi\rangle.$$

The latter is exactly (8) and so u is a weak solution of the given ODE (7).

The function u that satisfies equation (8) lies in H. To show that u is actually *twice differentiable* requires more work (known as regularity of solutions), along with the assumptions on b and the RHS f of (7).