UNBC UNIVERSITY OF NORTHERN BRITISH COLUMBIA

Department of Mathematics & Statistics

Math 402/602 Final Examination
Winter 2022
Submit (Online) by April 9, 9 pm.
Total mark: 100

Instructions. *Your work is to be submitted by Saturday, April 9, 9:00 PM on blackboard, as a single pdf file (scanned or typed or handwritten on a tablet).*

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1. Find the following quantities (showing the details leading to your answer)

(10) (a) $\min\left\{\int_{-\pi}^{\pi} |x - a\sin(x) - b\cos(x) - c\cos(7x)|^2 \,\mathrm{d}x, \quad a, b, c \in \mathbb{R}\right\}.$

(10) (b) $\max_{f\in L^2(0,\pi),\ \|f\|_{L^2}=1} \int_0^{\pi} x\, f(x)\,\mathrm{d}x.$

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2. Let T be the map defined on $L^2(0,\pi)$ as follows:

$$\forall f \in L^2(0,\pi),\ Tf := \int_0^{\pi} x f(x)\,\mathrm{d}x.$$

(5) (a) Verify that T is a bounded linear functional on $L^2(0,\pi)$.

(10) (b) Compute the operator norm $\|T\|$.

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3. The *Plancherel formula* reads

$$\forall\, f \in L^2(\mathbb{R}), \quad \int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x = \int_{-\infty}^{\infty} |\hat{f}(k)|^2\,\mathrm{d}k. \tag{1}$$

(you are **not** required to prove this formula).

For $a > 0$, the even exponentially decaying pulse is the function

$$f_e(x) := e^{-a|x|}.$$

The Fourier transform of the function f_e is given by

$$\hat{f}_e(k) = \sqrt{\frac{2}{\pi}}\frac{a}{k^2 + a^2}.$$

(10) (a) Apply the Plancherel formula to the even decaying pulse to evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^2}$.

(5) (b) How would you compute the integral in part (a) using elementary calculus?

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4. This problem has two parts:

(10) (a) Let V be a **complex** inner product space. Prove that, for all $f,\ g \in V$ we have

$$\langle f, g\rangle = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2 + \mathrm{i}\|f+\mathrm{i}g\|^2 - \mathrm{i}\|f-\mathrm{i}g\|^2\right). \tag{2}$$

(10) (b) The Parseval's formula for functions in $L^2(\mathbb{R})$ is

$$\int_{-\infty}^{\infty} f(x)g(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \hat{f}(k)\hat{g}(k)\,\mathrm{d}k. \tag{3}$$

Prove (3) using Plancherel's formula, which was announced in equation (1) above. **Hint**: Use the identity (2) in Part (a) of this problem.

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(10) 5. $B(X,Y)$ is our notation the space of *bounded* linear maps from X to Y.

Let X and Y be normed linear spaces such that X is finite-dimensional. Show that every linear mapping $T : X \to Y$ must belong to $B(X,Y)$ (i.e. must be a *bounded* linear map).

Note: it may be of help to use the fact that "all norms are equivalent on a finite dimensional space" in your proof.

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(10) 6. Let X and Y be normed linear spaces with X infinite-dimensional. Show that there must exist a linear mapping $T : X \to Y$ that does not belong to $B(X,Y)$.

Help: Since X is infinite dimensional, we have a countable linearly independent family $\{x_n\}_{n=1}^{\infty} \subset X$. Fix $y \neq 0$ in Y and define the map $T : X \to Y$ by $T\left(\dfrac{x_n}{\|x_n\|}\right) = ny$. Extend T linearly to X (explain how...) and use it for your proof.

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(10) 7. Let $X = C^\infty([0,1])$ consist of the smooth functions on $[0,1]$ that have continuous derivatives of all orders, equipped with the norm $\|\cdot\|_\infty$. The space X is a normed space, but it is not a Banach space, since it is incomplete.

Show that the differentiation operator $Du = u'$ is an **unbounded** linear map $D : X \to X$.

Hint: use, for example, the functions $\{u_\lambda := e^{\lambda x}\}_\lambda$ where the λ's are chosen appropriately.

TOTAL MARKS: 100