UNIVERSITY OF NORTHERN BRITISH COLUMBIA
Department of Mathematics & Statistics

Math 402/602, Midterm, Winter 2022
Submit (Online) by Monday March 7, 10:00 AM.
Total mark: 100

Instructions. *Your work is to be submitted by Monday March 7, 10:00 AM on blackboard, as a single pdf file (scanned or typed or handwritten on a tablet).*

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(10) 1. Let $(V, \|\cdot\|_1)$ be a complete vector space and let $\|\cdot\|_2$ be another norm on V . Recall that two norms $\|\cdot\|_1$ and $\|\cdot\|_1$ are called equivalent norms if there exist two constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1.$$

Prove that the normed space $(V, \|\cdot\|_2)$ is complete if and only if the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

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(10) 2. Show that if $x_k \to x^\star$ (as $k \to +\infty$) in a normed space V, then $\|x_k\| \to \|x^\star\|$ (in $\mathbb{R}$), as $k \to +\infty$.

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(5) 3. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed vector spaces, and $f : X \to Y$ be a continuous map. Prove that if $K \subset X$ is compact, then $f(K) \subset Y$ is also compact.

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4. Let $\{f_n\}_n$ be the sequence of functions defined by,

$$f_n(x) = ((x - 1/2)^2 + 1/n)^{1/2}, \quad \text{for all } 0 \leq x \leq 1, \quad n \in \mathbb{N}.$$

(5) (a) Does $f_n \in C^1([0, 1])$?

(10) (b) Show that $\{f_n\}$ converges uniformly, over $[0, 1]$, to a function f to be determined.

(5) (c) Use the above observations to decide whether $C^1([0, 1])$, equipped with the sup norm, is a Banach space or not.

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(15) 5. Recall: Cauchy-Schwarz inequality yields that, for any x and y in a Hilbert space H, we have

$$|\langle x, y\rangle| \leq \|x\|_H \cdot \|y\|_H ,$$

with strict inequality unless x and y are collinear, i.e., unless one of x or y is a multiple of the other. In light of the above statement, find the maximum and the maximizer of

$$\int_0^1 e^{-(1-t)} f(t)\, dt$$

on the closed unit ball of $L^2[0, 1]$.

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6. Suppose that $T : X \to X$ is a bounded linear operator on a normed vector space X. For $n = 1, 2, 3, \cdots$, the map T^n stands for

$$T^n := \underbrace{T \circ T \cdots \circ T}_{n-times}.$$

Note that if is T linear then T^n is linear. We will understand T^0 as the identity map Id or I from X into X.

For convenience, we denote by $\mathcal{B}(X, X)$ for the space of **bounded linear operators from X to X.**

(10) (a) Show that for all $n \in \mathbb{N}$, $\|T^n\| \leq \|T\|^n$, where $\|\cdot\|$ denotes the operator norm on $\mathcal{B}(X, X)$.

(5) (b) Consider the infinite series (Neumann Series)

$$\sum_{n=0}^{\infty} T^n = I + T + T^2 + \cdots$$

(this infinite sum is formal so far). Let $S_n := \sum_{k=0}^{n} T^k$ denote the finite partial sum. Note that, $S_n \in \mathcal{B}(X, X)$ (think why and how...).

Show that, if $\|T\| < 1$ then $I - T^n$ converges to Id in the space $(\mathcal{B}(X, X), \|\cdot\|)$.

(5) (c) Show that, if $\|T\| < 1$, then the Neumann Series converges in $\mathcal{B}(X, X)$ and

$$\lim_{n\to\infty} \|(I - T) \circ S_n - I\| = 0 = \lim_{n\to\infty} \|S_n \circ (\mathrm{Id} - T) - I\|.$$

(5) (d) Show that if $\|T\| < 1$ then $I - T$ is invertible and its inverse is the Neumann Series.

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(15) 7. Let $T : X \to Y$ be a bounded linear map between two Banach spaces X and Y. Suppose there exists $c > 0$ such that

$$\forall x \in X, \quad c\|x\|_X \leq \|Tx\|_Y.$$

Show that the range of T is a closed subspace of Y.

Hint. Let $\{y_n = Tx_n\}_n$ be a sequence in Range(T) that converges to y. Show that y must be in the range of T.

TOTAL MARKS: 100