Optimal initial data for an RDA equation

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Statement of the problem

$$\begin{cases} \partial_t u - \Delta u - q \cdot \nabla_x u = f(u) & \text{in } (0,T) \times \Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu}(t,x) = 0, & \text{for all } t \in (0,T) \text{ and } x \in \partial\Omega, \end{cases} \quad (1)$$

Consider the set of all initial populations of size m,

- is there an optimal distribution of individuals, at the initial time, that makes the total population at a time T maximal?
- if such an optimal initial datum exists, what about its uniqueness?
- More questions (later)

Assumptions

1. The reaction term f satisfies the following assumptions:

$$\begin{cases} f \in C^1(\Omega), \\ f' \text{ is Lipschitz-continuous there exists } M \text{ such that } |f'| \leq M, \\ f(0) = f(1) = 0. \end{cases} \tag{2}$$

2. The underlying advection $q(x) = (q_1(x), \ldots, q_N(x))$ is a vector field that satisfies

$$q \in C^{1,\delta}(\Omega), \quad \text{for some } \delta > 0, \qquad \text{and} \qquad q \cdot \nu = 0 \text{ on } \partial\Omega. \tag{3}$$

well-posedness

$$\begin{cases} \partial_t u - \Delta u - q \cdot \nabla_x u = f(u) & \text{in } (0,T) \times \Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu}(t,x) = 0, & \text{for all } t \in (0,T) \text{ and } x \in \partial\Omega, \end{cases} \tag{4}$$

where Ω is a bounded, connected and smooth domain of $\mathbb{R}^N$. At a prefixed a time $T > 0$, maximize the functional

$$\mathcal{I}_T(u_0) := \int_\Omega u(T,x)\,\mathrm{d}x \tag{5}$$

among all possible initial data $u_0 \in \mathcal{A}_m$, where

$$\mathcal{A}_m := \left\{ u_0 \in \mathcal{A},\ \int_\Omega u_0 = m \right\}, \quad \mathcal{A} := \left\{ u_0 \in L^1(\Omega),\ 0 \le u_0 \le 1 \right\}, \tag{6}$$

and $u(t,x)$ is the solution of (4) with an initial datum $u(0,\cdot) = u_0(\cdot)$.

$u \in \mathcal{A} \mapsto \mathcal{I}_T(u)$ is bounded

- Since $0 \leq u_0(x) \leq 1$ and $f(0) = f(1) = 0$, it follows that $\bar{U} = 1$ is a supersolution of (4) and $\underline{U} = 0$ is a subsolution of (4).
- An application of the maximum principle allows us to conclude that $0 \leq u \leq 1$.
- The above explanation implies that the operator $\mathcal{I}_T : \mathcal{A} \to \mathbb{R}$, defined by

$$\mathcal{I}_T(u_0) = \int_\Omega u(T, x)\,\mathrm{d}x,$$

  where u is the solution of equation (4) assumes finite values, whenever the argument u_0 belongs to the family $\mathcal{A}$.
- Note that u_0 plays the role of an initial value for (4), which leads to the integrand of our functional $\mathcal{I}_T$.

Prior works

- The first question in our paper is addressed in Nadin and Marrero [2] in a more particular setting: for a reaction-diffusion model without an advection field. The results of Nadin *et al.* [2] prove the existence of an optimal initial datum in the case where $q \equiv 0$ in (4).
- They also announce the open question about the uniqueness of the maximizer. We answer the uniqueness question in Theorem 2 of this present work, even in the presence of a drift term.

Existence and Uniqueness of a Maximizer

Theorem (E., Abdul Halim 21, Existence of an optimal initial datum)

Let Ω be a bounded domain and let f and a satisfy the assumptions stated in (2) *and* (3) *respectively. Then, there exists $\bar{u}_0 \in \mathcal{A}_m$ such that*

$$\max_{u_0 \in \mathcal{A}_m} \mathcal{I}_T(u_0) = \mathcal{I}_T(\bar{u}_0).$$

The following result confirms the uniqueness of the optimal initial datum.

Theorem (Uniqueness of the optimal initial datum)

Suppose that f and a satisfy the assumptions (2) *and* (3) *respectively. Furthermore, assume that f is strictly concave. Then, the optimal initial datum $\bar{u}_0$, obtained in Theorem 1, is unique.*

influence of advection on $\max_{u_0 \in \mathcal{A}} \mathcal{I}_T(u)$

Another natural question is

- *whether the presence of an advection term influences the value of the optimal mass $\int_\Omega u(T, x)\,\mathrm{d}x$ or not.*
- *Our second goal is to compare the optimal total mass in the case where an advection is present to the optimal total mass in the case where no advection is considered in the model. Does the addition of an advection to the medium enhance the total mass at a time T?*

Under an additional assumption on the divergence of the advection field, we show that the total mass at time T in model (7) is larger than the maximal total mass in model (8), even when the reaction-diffusion equation in (8) has an optimal initial datum.

Setting

For $U_A^0 \in \mathcal{A}_m$, let $\mathcal{I}_T^A(U^0) = \int_\Omega U_A(T,x)\,\mathrm{d}x$, where U_A is the solution of

$$\begin{cases} \partial_t U_A - \sigma \Delta U_A - Aq(x) \cdot \nabla_x U_A = f(U_A) & \text{in } (0,T) \times \Omega \\ U_A(0,x) = U_0(x) & \text{in } \Omega, \\ \frac{\partial U_A}{\partial \nu}(t,x) = 0 & t \in (0,T),\ x \in \partial\Omega. \end{cases} \tag{7}$$

Let $\mathcal{I}_T(V_0) = \int_\Omega V(t,x)\,\mathrm{d}x$, where V is the solution of

$$\begin{cases} \partial_t V - \sigma \Delta V = f(V) & \text{in } (0,T) \times \Omega, \\ V(0,x) = V_0(x) & \text{in } \Omega, \\ \frac{\partial V}{\partial \nu}(t,x) = 0 & t \in (0,T),\ x \in \partial\Omega. \end{cases} \tag{8}$$

Negative divergence drifts are enhancing

Theorem (Enhancement of the total mass by advection)

We assume that f *and* a *satisfy* (2) *and* (3) *respectively. Moreover, we assume that*

$$\alpha := \max_{x \in \overline{\Omega}} \nabla \cdot q(x) < 0 \tag{9}$$

and

$$f(s) \geq 0, \quad \text{for all} \quad s \in [0, 1]. \tag{10}$$

If

$$A \geq -\frac{M|\Omega|}{m\alpha} > 0, \tag{11}$$

we then have

$$\inf_{u_0 \in \mathcal{A}_m} \mathcal{I}_T^A(u_0) \geq \max_{V_0 \in \mathcal{A}_m} \mathcal{I}_T(V_0). \tag{12}$$

On the proof of existence

Lemma

Under the assumptions (2) and (3), the solution $u = u(t,x)$ of (4) satisfies the following properties:

$$u \in L^2\left(0, T; H^1(\Omega)\right) \cap L^\infty\left(0, T; L^2(\Omega)\right) \text{ and} \tag{13}$$

$$\partial_t u \in L^2\left(0, T; H^{-1}(\Omega)\right). \tag{14}$$

- Maximum principle implies that $\mathcal{I}_T$ is bounded over $\mathcal{A}_m$.
- So $\sup\limits_{\mathcal{A}_m} \mathcal{I}_T(\mathcal{A}_m)$ exists.
- Consider a maximizing sequence U_0^n in the set $\mathcal{A}_m$, such that

$$\lim_{n\to\infty} \mathcal{I}_T(U_0^n) = \sup_{U_0 \in \mathcal{A}_m} \mathcal{I}_T(U_0).$$

Existence proof (continued)

- As $U_0^n \in L^\infty(\Omega)$, there exists $\bar{U}_0 \in L^\infty(\Omega)$ such that U_0^n converges to $\bar{U}_0$ in the weak $\star$ topology. For $\varphi = 1 \in L^1(\Omega)$, we get

$$\int_\Omega U_0^n \varphi = \int_\Omega U_0^n = m \to \int_\Omega \bar{U}_0 \varphi = \int_\Omega \bar{U}_0 = m, \quad \text{as } n \to +\infty.$$

  Hence, $\bar{U}_0 \in \mathcal{A}_m$.
- Let U^n be the weak solution of (4), where the initial values is taken as U_0^n.
- We prove that $\{U^n(T, \cdot)\}$ is bounded in $L^2(\Omega)$.
- Lemma $\implies$ U^n converges, in the weak $\star$ topology, to U in $L^\infty\left(0, T; L^2(\Omega)\right)$.
- Aubin-Lions Lemma [3] leads to: $\{U^n\}_n$ converges strongly to U in $L^2\left([0, T]; L^2(\Omega)\right)$.

Existence proof (continued)

- Eventually one gets that the weak limit U of the maximizing sequence U_n is a weak solution of (4).
- testing on $\varphi = 1$ yields

$$\sup_{\mathcal{A}_m} \mathcal{I}_T(u_0) = \lim_{n\to\infty} \mathcal{I}_T(U_0^n) = \lim_{n\to\infty} \int_\Omega U^n(T,x)\,\mathrm{d}x$$
$$= \lim_{n\to\infty} \int_\Omega U^n(T,x)\varphi(x)\,\mathrm{d}x = \int_\Omega U(T,x)\,\mathrm{d}x = \mathcal{I}_T(\bar{U}_0).$$

- Therefore, $\bar{U}_0$ is a maximizing element of $\mathcal{I}_T$ in $\mathcal{A}_m$.

Uniqueness' proof

- First prove that $\mathcal{I}_T$ is strictly concave: Let $\lambda \in (0,1)$ and let u be the solution of (4) with I.C. $\lambda U_1^0(x) + (1-\lambda)U_2^0(x)$ and let $v(t,x) = \lambda U_1(t,x) + (1-\lambda)U_2(t,x)$ where U_1, U_2 are the solutions of (4) with initial condition $U_1^0(x)$, $U_2^0(x)$ respectively...
- The above claim follows (after a non-short argument) from the maximum principle and the fact that f is concave.
- Suppose that V_1^0 and V_2^0 in $\mathcal{A}_m$, are two maximizers of $\mathcal{I}_T$ in $\mathcal{A}_m$.
- Since $\mathcal{I}_T$ is strictly concave, it follows that

$$\mathcal{I}_T(\mu V_1^0 + (1-\mu)V_2^0) > \mu \mathcal{I}_T(V_1^0) + (1-\mu)\mathcal{I}_T(V_2^0) = \sup_{u_0 \in \mathcal{A}_m} \mathcal{I}_T(u_0).$$

However, this contradicts the fact that V_1^0 and V_2^0 are maximizers of $\mathcal{I}_T$ in $\mathcal{A}_m$. Therefore, the maximizer of $\mathcal{I}_T$ is unique in the set $\mathcal{A}_m$.

References

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