Exercise 1.

Consider

$$(P)_\lambda \qquad \begin{cases} -\Delta u &= \lambda f(u) \quad \Omega \\ u &= 0 \qquad \partial\Omega, \end{cases} \tag{1}$$

where f is smooth, increasing, convex and superlinear at ∞ (i.e. $\lim_{t\to\infty} \frac{f(t)}{t} = \infty$) and where $f(0) = 1$, Ω is a bounded domain in $\mathbb{R}^N$, and $\lambda > 0$ is a parameter.
Prove that there exists some $\lambda^* \in (0, \infty)$ (the extremal parameter) such that for all $0 < \lambda < \lambda^*$ there exists a smooth solution u of $(P)_\lambda$. Moreover, for all $\lambda > \lambda^*$ there are no weak solutions of $(P)_\lambda$ (here one needs to define weak).

Define

$$\lambda^* := \sup \{0 \le \lambda : (P)_\lambda \text{ has a smooth solution}\}.$$

We first show $\lambda^* < \infty$. Let $\phi > 0$ denote the first eigenvalue of $-\Delta$ in H_0^1. So $-\Delta\phi = \lambda_1\phi$ in Ω. Let u be smooth solution of $(P)_\lambda$. Multiply $(P)_\lambda$ by ϕ and integrate by parts to see that

$$\int_\Omega (\lambda f(u) - \lambda_1 u)\, \phi(x) dx = 0$$

and since $\phi > 0$ we must have

$$\inf_\Omega (\lambda f(u) - \lambda_1 u) = 0.$$

So there is some $t \ge 0$ (recall $\lambda f(u) \ge 0$ so $u \ge 0$) such that $\lambda f(t) - \lambda_1 t = 0$. Hence we have

$$\lambda \le \lambda_1 \sup_{t\ge 0} \frac{t}{f(t)}.$$

So $\lambda^* \le \lambda_1 \sup_{t\ge 0} \frac{t}{f(t)}$ which is finite after considering fact f superlinear at ∞.

We now show that $\lambda^* > 0$. This follows directly from the implicit function theorem applied to the solution $\lambda = 0$, $u = 0$. Instead of using the IFT we can use a sub/super solution approach. Clearly $\underline{u} = 0$ is a subsolution. We just need to find a supersolution $\overline{u} \ge 0$. Let $-\Delta\overline{u} = 1$ in Ω with $\overline{u} = 0$ on $\partial\Omega$. Then $\overline{u} > 0$ in Ω and let $M := \max_\Omega \overline{u}$. Then $\overline{u}$ a supersolution provided $1 \ge \lambda f(\overline{u})$ in Ω and it is sufficient that $1 \ge \lambda f(M)$. But taking $\lambda > 0$ small enough ensures this.

We now show that for all $0 < \lambda < \lambda^*$ there exists a smooth solution. Fix $0 < \lambda < \lambda^*$ and let $\lambda \le \gamma \le \lambda^*$ is such that there is a smooth solution v of $(P)_\gamma$ (this exists by the definition of λ^* and supremum). Then to solve $(P)_\lambda$ we apply the sub/supersolution proposition with $\underline{u} = 0$ and $\overline{u} = v$ to find a smooth solution $0 \le u \le v$ in Ω of $(P)_\lambda$. Lets check that $\overline{u} := v$ is indeed a supersolution. Its nonnegative on $\partial\Omega$ and

$$-\Delta v = \gamma f(v) \ge \lambda f(v) \qquad \Omega$$

and so it is a supersolution.

To complete the proof of 1) we need to show there are no weak solutions of $(P)_\lambda$ for any $\lambda > 0$.
Idea. Suppose there is some sort of weak solution w of $(P)_\lambda$ for some $\lambda^* < \lambda$. Fix $\lambda^* < \lambda_1 < \lambda$.

The idea is to use w to construct a bounded supersolution v of $(P)_{\lambda_1}$ and then we could apply the sub/supersolution method to see that $\lambda^* \geq \lambda_1$ (recall λ^* is the supremum of λ with smooth solutions) to get a contradiction. The form of v we look for is $v = g(w)$ where g is some bounded function. For v to be supersolution of $(P)_{\lambda_1}$ it is sufficient that $v \geq 0$ on $\partial\Omega$ and

$$g'(w)\lambda f(w) - g''(w)|\nabla w|^2 \geq \lambda_1 f(g(w)) \qquad \Omega$$

(here we just wrote out $-\Delta g(w) \geq \lambda_1 f(g(w))$.) So we want g bounded, $g \geq 0$ on $[0, \infty)$ and assume that $g''(w) \leq 0$ on $w \in [0, \infty)$. Then it is sufficient that

$$\lambda g'(w) f(w) \geq \lambda_1 f(g(w)) \qquad \Omega.$$

We now switch to a particular case since its more transparent. Assume $f(t) = e^t$. So we want

$$\lambda g'(w) e^w \geq \lambda_1 e^{g(w)}.$$

Solve for g assuming we have equality and one sees we want g of the form

$$g(w) := \ln\left(\frac{1}{C + \frac{\lambda_1}{\lambda} e^{-w}}\right)$$

and then lets assume we want $g(0) = 0$. So one ends up with

$$g(w) = \ln\left(\frac{\lambda}{\lambda - \lambda_1 + \lambda_1 e^{-w}}\right)$$

and then we check that g satisfies all the requirements, ie. $g \geq 0$ on $(0, \infty)$, $g'' \leq 0$ on $(0, \infty)$, g bounded.