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(13) 1. Let $a : \Omega \to \mathbb{R}$ be a smooth function defined over a bounded smooth domain $\Omega \subset \mathbb{R}^n$. Show that the boundary value problem

$$\begin{cases} \nabla \cdot (a(x)\nabla u) + b(x)u = \lambda u, & x \in \Omega \\ u = 0, \quad x \in \partial\Omega. \end{cases} \tag{1}$$

has only trivial solution with $\lambda > 0$, when $b(x) \leq 0$ and $a(x) > 0$ in Ω.

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2. Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and consider

$$\begin{cases} u_t = \Delta u, & x \in \Omega, \quad t > 0 \\ u(0, x) = f(x). \end{cases} \tag{2}$$

with the boundary conditions

$$\frac{\partial u}{\partial \nu}(x) = 0 \text{ for } x \in \partial\Omega.$$

(10) (a) Show, with all necessary justifications, that the quantity

$$N(t) := \int_\Omega (u(t, x, y))^2 \, \mathrm{d}x\mathrm{d}y$$

is decreasing in t provided that f is not a constant function.

(5) (b) Show that $\dfrac{dN}{dt} \equiv 0$ when f is a constant function.

(10) (c) Deduce that, if problem (2) has a solution, then such solution is unique.

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(20) 3. Assume $f : \mathbb{R} \to \mathbb{R}$ is a C^1 **bounded** function with

$$f(0) = 0, \quad f'(0) > \lambda_1,$$

where λ_1 denotes the principal eigenvalue of $-\Delta$ with Dirichlet boundary conditions.

Use the method of sub- and supersolutions to show that there exists asolution u of

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega. \end{cases} \tag{3}$$

Help. Find a subsolution to (3) of the form $\varepsilon\varphi$, where ε is a constant to be chosen appropriately.

For a supersolution, find a suitable constant M so that the solution to $\begin{cases} -\Delta \bar{u} = M & \text{in } \Omega \\ \bar{u} = 1 & \text{on } \partial\Omega \end{cases}$ can be a supersolution to (3).

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4. Let $u(t,x) = v\left(\frac{x^2}{t}\right)$ for $x \in \mathbb{R}$ and $t > 0$.

(15) (a) Show that

$$u_t = u_{xx}$$

if and only if

$$(*) \qquad 4zv''(z) + (2+z)v'(z) = 0, \quad z > 0.$$

(10) (b) Show that the general solution of $(*)$ is

$$v(z) = c\int_0^z e^{-s/4}s^{-1/2}\mathrm{d}s + k,$$

where c and k are constants.

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5. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $v \in C^2(\overline{\Omega})$ satisfy

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega \\ u = g(x) & \text{on } \partial\Omega. \end{cases} \tag{4}$$

The goal of this problem is to prove that there exists a constant C, only depending on Ω, such that

$$\max_{\overline{\Omega}} |v| \le C(\max_{\partial\Omega} |g| + \max_{\overline{\Omega}} |f|). \tag{5}$$

The following questions are steps that lead to the mentioned statement.

(5) (a) Denote by $L := \max_{\overline{\Omega}} |f|$. Let

$$\hat{v}(x) := v(x) + \frac{L}{2n}\|x\|^2.$$

Show that

$$-\Delta\hat{v} \le 0 \text{ in } \Omega. \tag{6}$$

(5) (b) Why does it then follow that $\max_{\overline{\Omega}} |\hat{v}| \le \max_{\partial\Omega} |\hat{v}|$?

(2) (c) Compare $\max_{\overline{\Omega}} |v|$ to $\max_{\overline{\Omega}} |\hat{v}|$.

(5) (d) Deduce from the above observations that (5) holds.

TOTAL MARKS: 100