Submission of your homework assignment will be via Blackboard.

Exercise 1.

Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary. Let L be the elliptic operator given by

$$Lu := -\sum_{1\le i,j\le n} a_{ij}(x)\partial_{x_i x_j}u + \sum_{1\le i\le n} b_i \partial_{x_i}u + c(x)u,$$

where $(a_{ij})_{i,j}$ is a positive definite symmetric matrix, with smooth entries.

We say that an elliptic operator L satisfies the weak maximum principle if the following statement holds:

whenever $Lw \ge 0$ in Ω and $w \ge 0$ on $\partial\Omega$, we must have $w \ge 0$ everywhere in Ω.

Recall that, when $c \ge 0$, the weak maximum principle holds for the operator L. In this problem, we will derive a necessary and sufficient condition so that the weak max. principle holds for such class of operators (without an assumption on the sign of c).

Prove the following

Theorem. Suppose that λ is the principal eigenvalue of L with Dirichlet boundary conditions. That is, there exists $\varphi > 0$ such that

$$\begin{cases} L\varphi = \lambda\varphi \text{ in } \Omega \\ \varphi = 0 \text{ on } \partial\Omega. \end{cases}$$

Then L satisfies the weak maximum principle if and only if $\lambda < 0$.

Exercise 2.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Let $q : \Omega \to \mathbb{R}^n$ be a smooth vector field that satisfies the incompressibility condition

$$\nabla \cdot q \equiv 0 \text{ in } \Omega.$$

Consider the dynamical system

$$\begin{cases} \dfrac{\mathrm{d}\Phi}{\mathrm{d}t}(t,\mathbf{x}) = q(\Phi(t,\mathbf{x})), \quad t > 0,\ \mathbf{x} \in \Omega, \\ \\ \Phi(0,\mathbf{x}) = \mathbf{x}, \quad \mathbf{x} \in \Omega. \end{cases} \tag{1}$$

Let $J(t,\mathbf{x})$ be the determinant of the Jacobian matrix $\dfrac{\partial \Phi}{\partial \mathbf{x}}(t,\mathbf{x})$ (the general entry of this matrix is $\partial \Phi_i / \partial \mathbf{x}_j$).

1. Prove by direct calculations that

   $$\frac{\partial J}{\partial t} = (\nabla_{\mathbf{x}} \cdot q) J(t,x), \quad J(0,\mathbf{x}) = 1.$$

   Help. For simplicity, assume in this part that $n = 2$.

2. Conclude that $J \equiv 1$ in $[0,\infty) \times \Omega$

3. For each $t > 0$, let

   $$\Omega_t := \Phi(t,\Omega)$$

   be the transformation of the domain Ω under the flow q. Show that

   $$\text{measure}(\Omega_t) = \text{measure}(\Omega).$$