Submission of your homework assignment will be via Blackboard.

Exercise 1.

Let Ω be an open subset of $\mathbb{R}^2$. Let $u(x, y)$ be a nonconstant solution of the equation $u_{xy} = 0$. Is it possible for u to attain an interior maximum in Ω?

Exercise 2 (Boundedness of domain is crucial in the maximum principle)**.**

In this problem, we show that if one removes the assumption of Ω bounded then one can lose uniqueness of solutions. Consider

$$\begin{cases} u_{xx} + u_{yy} = 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where

$$\Omega := \{(x, y) \in \mathbb{R}^2 \text{ such that } x \in \mathbb{R} \text{ and } 0 < y < \pi\}.$$

Of course, $u = 0$ is a solution of (1). Find another solution of the form $u(x, y) = e^{\alpha x} \sin(\beta y)$ where α and β are to be determined.

Exercise 3.

Consider the *nonlinear* problem

$$\begin{cases} -\Delta u(x) + h(u(x)) = 0, \text{ for } x \in \Omega \\ \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2}$$

where Ω a bounded domain in $\mathbb{R}^n$ with smooth boundary and where $h : \mathbb{R} \to \mathbb{R}$ is smooth with $h'(z) \geq 0$ for all $z \in \mathbb{R}$. Show (2) has at most one solution.

Exercise 4.

Let $u : \mathbb{R}^n \to \mathbb{R}$ be a harmonic function such that

$$|u(x)| \leq C_1 + C_2|x|^\sigma,$$

for some $1 < \sigma < 2$ and some constants $C_{1,2}$. Show that

$$u(x) = \mathbf{a} \cdot x + \mathbf{b}$$

for some constant vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$.

Hint: show that $u_{x_i x_j}(x) = 0$ for all $x \in \mathbb{R}^n$.