An integro-difference model in a patchy landscape

CMS - 2021 Summer Meeting

Mohammad El Smaily [1]

UNIVERSITY OF NORTHERN BRITISH COLUMBIA

Ottawa, June 8, 2021

[1] NSERC CRSNG I acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC), NSERC-Discovery Program.

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4. Proof ideas

The model

We consider the integral equation

$$u_{n+1}(x) = \int_\Omega k(x, y) F(u_n(y)) \, \mathrm{d}y, \tag{1}$$

- $u_n(x)$ $(n = 1, 2, \ldots)$ stands for the density of the population in the n^{th} generation at a location x. The population density in the nth generation, at a location x, is denoted by $u_n(x)$.
- The growth phase is described by the function $F \geq 0$.
- The dispersal phase is described by a dispersal kernel $k(x, y)$. The probability that an individual, who started its dispersal process at x, will settle in $[y, y + dy)$ is then given by the product $k(x, y)dy$.
- Density in the next generation, at a location x, is obtained by summing up arrivals at x from all possible locations y. This yields the integral equation (1).

Assumptions: $u_{n+1}(x) = \int_\Omega k(x,y)F(u_n(y))\,\mathrm{d}y$

We consider the case $\Omega = (-a, a)$.

1. The function $k \geq 0$ is a bounded nonnegative function on Ω:

$$\text{For all } (x,y) \in \Omega,\ 0 < \delta < k(x,y) \leq \Lambda. \tag{2}$$

2. k is continuous with respect to each variable except at a finite set of points $\{a_i\}_{1\leq i\leq n} \subset \Omega$. This divides Ω into intervals

$$\Omega_i = (a_i, a_{i+1}) \text{ for } 1 \leq i \leq n-1,\quad \Omega_0 = (-a, a_1) \text{ and } \Omega_n = (a_n, a).$$

3. F is continuous, $F \geq 0$, $F \in L^\infty(\Omega)$, and $\|F\|_{L^\infty(\Omega)} \leq M$.
4. The function F is strictly increasing on $[0, \infty)$ and it vanishes elsewhere. We assume that F differentiable at 0 and we set

$$r_0 := F'(0) > 1. \tag{3}$$

5. F satisfies

$$\frac{F(u)}{u} < \frac{F(v)}{v},\quad \text{for all } u > v > 0. \tag{4}$$

Prior works

- Lutscher et al. 2014 [4]: a discontinuity in the dispersal kernel k at the interface of these patches appears in the study done in [4]. In , it is shown that the dispersal kernel can be characterized as the Green's function of a second-order differential operator.
- Watmough and Beykzadeh [1] included different dispersal rates in each patch as well as different degrees of bias at the patch boundaries.
- Lewis et al. [3], which considers the same model (1), but with a different set of assumptions on the kernel k. In [3], the kernel k is assumed to be in the form $k(x, y) = k(x - y)$. The more important difference between our work and [3] is that the kernel k is assumed to be continuous in [3].
- The discontinuity points account for a different dispersal behaviour; hence, a change in the patch where the population is moving. The discontinuity of k adds more technicality to the proofs of the main results, especially those involving comparison arguments.

Analogue of RDA equations

- The model $u_{n+1}(x) = \int_{\Omega} k(x,y) F(u_n(y))\, \mathrm{d}y$ is a **discrete**-time analogue of the Reaction-Advection-Diffusion equations in population dynamics.
- RDA eqs are semilinear parabolic partial differential equations

$$\frac{\partial u}{\partial t}(t,x) = \Delta u + q(x) \cdot \nabla u + f(u), \quad t > 0, \quad x \in \mathbb{R}^N, \tag{5}$$

for some incompressible vector field $q : \mathbb{R}^N \to \mathbb{R}^N$.
- When (5) is used to describe the evolution of a population density $u(t,x)$, it is assumed that the dispersal of the species follows the normal distribution with zero mean, which however is not the case for most species [Lewis (1997)].
- Traveling fronts, or pulsating traveling fronts, form a particular class of solutions to such equations.

$\mathcal{T}$ linearized at $u = 0$

We denote the linearization of the nonlinear operator $\mathcal{T}$, at $u = 0$, by

$$\mathcal{T}_0(u)(x) := r_0 \int_\Omega k(x,y)u(y)dy, \; x \in \Omega.$$

We denote by

$$X = \left\{ u \in L^2(\Omega),\, u \text{ is continuous on } \Omega \text{ except at finitely many points} \right\}. \qquad (6)$$

Theorem (Krein-Rutman Theorem)

Let E be a Banach space and let $K \subset E$ be a closed convex cone such that $K - K$ is dense in E and $K^\circ \neq \phi$. Let $\mathcal{A} : E \to E$ be a non-zero compact linear operator which is strictly positive, meaning that $\mathcal{A}(K \setminus \{0\}) \subset K^\circ$. Then, there exists a principal eigenvalue $\lambda_0 = r(\mathcal{A}) > 0$ of $\mathcal{A}$, and an eigenfunction $u \in K^\circ$ of $\mathcal{A}$ such that $\mathcal{A}u = \lambda_0 u$.

Existence of a principal eigenpair

We will apply Theorem 1 to the operator $\mathcal{T}_0$, where we let the Banach space E be $L^2(\Omega)$ and take

$$K = L^2(\Omega)^+ \cap C_p(\Omega)^+,$$

where

$$L^2(\Omega)^+ := \{u \in L^2(\Omega), \text{ such that } u \geq 0 \text{ in } \Omega\}$$

and $C_p(\Omega)^+$ is the set of positive functions that are piecewise continuous on Ω. It follows that

$$K^\circ = K^{++} = \{u \in K, \text{ such that } u > 0 \text{ in } \Omega\}.$$

Proposition

The operator $\mathcal{T}_0$ maps $L^2(\Omega)$ into itself. Moreover, $\mathcal{T}_0$ is a strictly positive compact operator and there exists an eigenfunction $\phi_0 \in K^\circ = K^{++}$ of $\mathcal{T}_0$ that corresponds to the principal eigenvalue $\lambda_0 := r(\mathcal{T}_0)$.

Extinction: $F(0) = 0$ and $\lambda_0 \leq 1$

Theorem (E.-Abdul Halim)

Let $\{u_n\}_n$ be a solution of (1), with the initial condition $u_0 \in X$. Suppose that F and k satisfy the assumptions 1, 3 and 4. Assume furthermore that $F(0) = 0$. If $\lambda_0 \leq 1$, then

1. *0 is the only stationary solution of (1) in X.*
2. *The sequence $\{u_n\}_n$ converges to 0 in $L^2(\Omega)$, as $n \to \infty$.*

The proof of this theorem is lengthy. The result is similar to what happens in the case of a continuous kernel of the form $k(x, y) = k(x - y)$, which was considered in Lewis et al. 2018.

Persistence: $F(0) = 0$ and $\lambda_0 \geq 1$

Theorem (E.-Abdul Halim)

Suppose that F and k satisfy the assumptions 1, 3 and 4. Assume furthermore that $F(0) = 0$. Suppose that $u_0 \in X$, $u_0 \geq 0$, $u_0 \not\equiv 0$ and that $\lambda_0 > 1$. Then,

1. *there exists a unique positive stationary solution w of (1).*
2. *The sequence $\{u_n\}_n$ converges to w in $L^2(\Omega)$, as $n \to \infty$.*

In the next theorem, we will see that the stationary state will be positive, regardless of λ_0, whenever $F(0) > 0$.

Theorem

Suppose that F and k satisfy the assumptions 1, 3 and 4. Assume furthermore that $F(0) > 0$. If $u_0 \in X$, $u_0 \geq 0$ and $u_0 \not\equiv 0$, then

1. *there exists a unique positive stationary solution w of (1).*
2. *The sequence $\{u_n\}_n$ converges to w in $L^2(\Omega)$, as $n \to \infty$.*

A comparison principle for IDE type models

$$X = \left\{ u \in L^2(\Omega),\ u \text{ is continuous on } \Omega \text{ except at finitely many points} \right\}. \tag{7}$$

Lemma

Let $u, v \in X$ (defined in (7)), such that $u \geq \mathcal{T}(u)$ and $v \leq \mathcal{T}(v)$. If $u > 0$, then $u \geq v$.

Lemma

If $u_1 \geq u_0$ (respectively $u_1 \leq u_0$), then $\{u_n := \mathcal{T}(u_{n-1}\}_n$ is increasing (respectively decreasing).

Sketch of the Proof of Theorem 5 (case $F(0) > 0$)

Here, we have the assumption that $F(0) > 0$. We fix $N > 2a\Lambda M$ and note that

$$\mathcal{T}(N) = \int_{-a}^{a} k(x, y)F(N)\,\mathrm{d}y \leq 2a\Lambda M < N.$$

Lemma 7 then yields that $\{\mathcal{T}^n(N)\}$ is decreasing. Since $\{\mathcal{T}^n(N)\}$ is bounded below by 0 uniformly, there exists $w \in X$, such that $\{\mathcal{T}^n(N)\}_n$ converges pointwise to w. That is, w is stationary solution of $\mathcal{T}$. We also show that w is the unique stationary solution of (1). We have

$$\mathcal{T}(0)(x) = \int_{\Omega} k(x, y)F(0)\,dy \geq 2a\delta F(0) > 0 \text{ for all } x \in \Omega.$$

Since $\{\mathcal{T}^n(0)\}_n$ is uniformly bounded above by N, one obtains $w_0 \in X$,[2] such that $\{\mathcal{T}^n(N)\}_n$ converges pointwise to w_0. This means that w_0 is stationary solution of $\mathcal{T}$.

An extra work shows that $w_0 = w$ and $\{u_n\}_n$ converges to w in $L^2(\Omega)$.

[2] Proving that $w_0 \in X$ turns out to be technical

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