Homogenization and fragmentation (reactive-diffusive media)

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Outline

Model 1

- Let's start with a homogeneous reaction-diffusion

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + u\,(\mu - \nu u),\ \ t > 0,\ \ x \in \mathbb{R}. \tag{1}$$

- An extension of above model to heterogeneous environments is the Shigesada-Kawasaki-Teramoto (1986) model (SKT, from now on)

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a_L(x)\frac{\partial u}{\partial x}\right) + u\,(\mu_L(x) - \nu_L(x)u),\ \ t > 0,\ \ x \in \mathbb{R}, \tag{1}$$

where the coefficients depend on the space variable x in a $L-$periodic fashion.

Model (Generalized)

Persistance in an infinite Reactive-Diffusive Habitat

Consider the reaction-diffusion model

$$\partial_t u = \partial_x \left(a_L(x)\partial_x u\right) + f_L(x,u),\ t \in \mathbb{R},\ x \in \mathbb{R},$$

$$0 \le u(t,x) \le p_L(x),\ \lim_{x\to-\infty} u(t,x) = 0, \lim_{x\to+\infty} u(t,x) - p_L(x) = 0,$$

a and f are 1-periodic in the spatial variable x.

- Q1: The domain ($\mathbb{R}$) is periodic (any period L). What scale/period L for coefficients slows down/speeds up an occurring invasion (larger scales, or smaller scales)?
- Q2: If an environment is composed of "habitat" and "non-habitat" **patches**, analyze the dependence of the speed on habitat fragmentation

Assumptions

Set $\mu(x) := \lim_{s \to 0^+} f(x,s)/s, \; \mu_L(x) := \mu\left(\frac{x}{L}\right).$

- We assume that
$$\left\| \begin{array}{l} \forall\, x \in \mathbb{R}, \; f(x,0) = 0, \\ \exists\, M \geq 0, \; \forall\, s \geq M, \; \forall\, x \in \mathbb{R}, \quad f(x,s) \leq 0, \\ \forall\, x \in \mathbb{R}, \quad s \mapsto f(x,s)/s \text{ is decreasing in } s > 0. \\ \int_0^1 \mu(x)dx > 0 \end{array} \right.$$

some interpretations:

- The growth rate μ is allowed to be positive in some regions (favourable regions) and negative in others (unfavourable regions).
- For each L, there is a stationary state $p_L(x)$ satisfying the elliptic PDE

$$\frac{\partial}{\partial x}\left(a_L(x)\frac{\partial p}{\partial x}\right) + f_L(x,p) = 0, \; x \in \mathbb{R}.$$

Featured solutions: pulsating traveling fronts

Berestycki, Hamel, Roques 2005: Consider the linear operator

$$\mathcal{L}_0 : \ \Phi \mapsto -(a_L(x)\Phi')' - \mu_L(x)\Phi, \text{ with periodicity conditions}$$

Stationary state $p_L(x)$ exists **if and only if** the principal eigenvalue $\rho_{1,L} < 0$.

- In our case, $\int_0^1 \mu(x) > 0$ [thus existence of nontrivial p_L]

$$\Longrightarrow \text{ PTFs: } u(t,x) = q(x-ct,x)$$

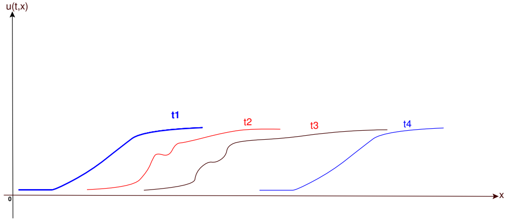


Figure: $p_L \equiv 1$, pulsations in the front are just from $a(x)$ and $f(x,\cdot)$)

More precisely

[Pulsating traveling fronts] $u = u(t,x)$ is called a pulsating traveling front propagating from right to left with an effective speed $c \neq 0$ (> 0 here) if u is a classical solution of:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(a_L(x)\dfrac{\partial u}{\partial x}\right) + f_L(x,u), \quad t \in \mathbb{R},\ x \in \mathbb{R}, \\ \forall\, k \in \mathbb{Z},\ \forall\,(t,x) \in \mathbb{R}\times\mathbb{R}, \quad u(t+\dfrac{kL}{c},x) = u(t,x+kL), \\ 0 \leq u(t,x) \leq p_L(x), \\ \lim\limits_{x\to-\infty} u(t,x) = 0 \text{ and } \lim\limits_{x\to+\infty} u(t,x) - p_L(x) = 0, \end{cases} \tag{2}$$

where the above limits hold locally in t.

Theorem (Minimal speed, traveling fronts, (BHR 2005))

$\exists c_L^* = c_{a,f}^* > 0$ *such that a PTF with a speed* c *exists if and only if* $c \geq c^*$.

the homogenized speed

Theorem (El Smaily, Hamel & Roques, DCDS A)

Let c_L^* be the minimal speed of propagation corresponding to (2).

$$\lim_{L\to 0^+} c_L^* = 2\sqrt{<a>_H <\mu>_A}, \tag{3}$$

$<\mu>_A = \int_0^1 \mu(x)dx$ *(arithmetic mean of μ),*

$<a>_H = \left(\int_0^1 \frac{1}{a(x)}dx\right)^{-1} = <a^{-1}>_A^{-1}$ *(harmonic mean of diffusion)*

Observation: the "effective speed" (i.e. the limit) is the KPP speed for the homogeneous R.D eq

$$\bar{U}_t = < a >_H \bar{U}_{xx} + g(\bar{U}), \text{ with } g(\cdot) := \int_0^1 f(x,.)\, dx$$

near $L = 0$

Theorem (El Smaily, Hamel, Roques, DCDS A)

The map $L \mapsto c_L^*$ is of class C^∞ in an interval $(0, L_0)$ for some $L_0 > 0$ &

$$\lim_{L \to 0^+} \frac{dc_L^*}{dL} = 0, \quad \lim_{L \to 0^+} \frac{d^2 c_L^*}{dL^2} = \gamma \geq 0.$$

Moreover, $\gamma > 0$ IFF the function $\frac{\mu}{<\mu>_A} + \frac{<a>_H}{a}$ is not identically equal to 2.

So far:

- *We know the limit of the minimal speeds c_L^* as $L \to 0^+$ (the homogenization limit). Also the homogenized equation (separate work, E. 2011)*
- **Near the homogenization limit, the species tends to propagate faster when the spatial period of the environment is enlarged.**

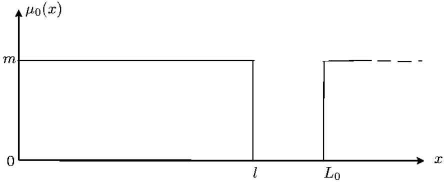
(both results) prove numerical conjectures

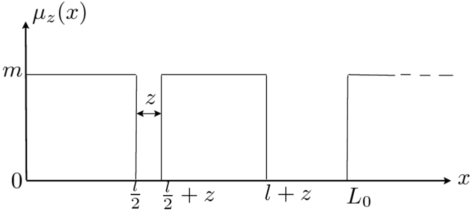
> *Spatial dynamics of invasion in sinusoidally varying environments, pp. 263-270.*

(N. Kinezaki, K. Kawasaki, N. Shigesada, Popul. Ecol. 48, 2006)

> *The above reference did draw conjectures (about the results in our Theorem 1) via a numerical study for particular diffusion a and (reaction) μ (sine functions)*

Influence of fragmentation on biological invasions



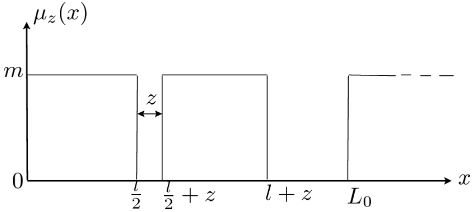


Fix a period $L_0 > 0$.
Assume that $a \equiv 1$, and that $\mu_{L_0} := \mu_z$ takes only the two values 0 and $m > 0$, and depends on a parameter $z > 0$ (the region $[0, L_0]$ is fragmented into two parts of same length $l/2$)

$$z \in (0, L_0 - l)$$

Regularity disclaimer: μ_z does not satisfy the general regularity assumptions. However, c_z^* can still be interpreted as the minimal speed of propagation of weak solutions, whose existence can be obtained by approaching μ_z with regular functions.

Habitat Fragmentation, at a fixed scale, slows down invasions



Theorem (E., Hamel, Roque, DCDS)

Let c_z^* be the minimal speed of propagation for (2), with $a_{L_0} \equiv 1$ and $\mu_{L_0} = \mu_z$. Assume that $l \in (3L_0/4, L_0)$. **Then $z \mapsto c_z^*$ is decreasing in $[0, (L_0 - l)/2]$, and increasing in $[(L_0 - l)/2, L_0 - l]$.**

Above shows that for z in $(0, L_0 - l)$, c_z^* is **larger when the minimal distance separating two habitat components is smaller**

The problem(s) in a variational language

- Rough Upper and lower bounds follow from

$$c_L^* = \min_{\lambda>0} \frac{k(\lambda, L)}{\lambda} = \frac{k(\lambda_L^*, L)}{\lambda_L^*}, \tag{4}$$

where $\lambda_L^* > 0$ and, for each $\lambda \in \mathbb{R}$ and $L > 0$, $k(\lambda, L)$ denotes the principal eigenvalue of the problem

$$\left(a_L \psi'_{\lambda,L}\right)' + 2\lambda a_L \psi'_{\lambda,L} + \lambda a'_L \psi_{\lambda,L} + \lambda^2 a_L \psi_{\lambda,L} + \mu_L \psi_{\lambda,L} = k(\lambda, L)\psi_{\lambda,L} \text{ in } \mathbb{R}, \tag{5}$$

- Finer bounds: $\varphi_{\lambda,L}(x) := e^{\lambda x}\psi_{\lambda,L}(x), \quad x \in \mathbb{R}$. Since $\psi_{\lambda,L}$ is unique up to multiplication, we can set

$$\int_0^2 \varphi^2_{\lambda,L}(x)dx = 1. \tag{6}$$

- The above choice ensures that

$$\int_0^2 \psi^2_{\lambda,L}(x)dx \leq \int_0^2 e^{2\lambda x}\psi^2_{\lambda,L}(x)dx = \int_0^2 \varphi^2_{\lambda,L}(x)dx = 1. \tag{7}$$

- the families $(\psi_{\lambda,L})_{\lambda,L}$ and $(\varphi_{\lambda,L})_{\lambda,L}$ remain bounded in $H^1(0,1)$ for L small enough and as long as λ stays bounded.

- $$\forall\ n, \quad \left(a_{L_n}\varphi_n'\right)' + \mu_{L_n}\varphi_n = k_n\varphi_n \text{ in } \mathbb{R} \quad (*)$$

- (very helpful) let $M_{L_n} = [1/L_n] + 1 \in \mathbb{N}$.

$$\exists\ \theta_n \in [0, L_n], \quad \psi_n(\theta_n) = \max_{x\in\mathbb{R}} \psi_n(x) = \max_{x\in[0,L_n]} \psi_n(x),$$

$\forall n \in \mathbb{N}, \quad \psi_n'(\theta_n) = 0.$

- Multiplying (*) by φ_n and IBP over $[\theta_n, \theta_n + M_{L_n} L_n]$, ...gives a sharp upper bound (after several convergence arguments on sequences of functions...)

- Proof of Theorem 2 contains lengthy computations.
$\dfrac{d^2c_L^*}{dL^2} \longrightarrow \dfrac{1}{12\lambda^*} \times \dfrac{\partial^4\tilde{k}}{\partial L^4}(\lambda^*, 0)$ as $L \to 0^+$
- The latter is computed through an integration by parts that splits into 45 integrals and converges eventually