Curved Fronts in a Shear Flow

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Outline

Problem in mathematical terms

Prior works on conical fronts

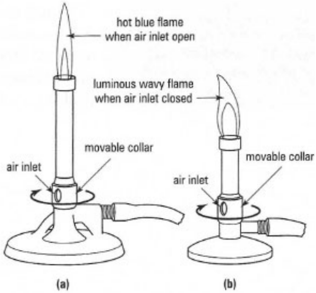
- Planar fronts

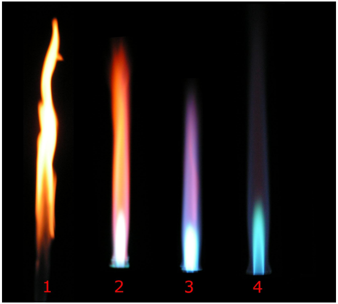
Curved fronts in the case of combustion nonlinearity

Sketch of the proof

Asymptotics in a shear flow with large amplitude

Setting





- A Bunsen burner, an air inlet with movable collar
- When the air inlet is open, lots of air mixes with burning fuel
- produces a "noisy, roaring" flame
- Can we speed-up the heating by using the Advection's (air-flow) influence?

The model

We will use the model

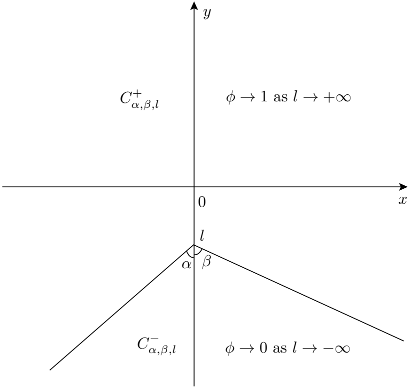
$$\frac{\partial u}{\partial t} = \Delta u + q(x)\frac{\partial u}{\partial y} + f(u), \text{ for all } t \in \mathbb{R},\ (x, y) \in \mathbb{R}^2, \tag{1}$$

where $u(t, x, y)$ stands for the "rescaled" temperature.

- The nonlinearity is to be discussed
- Prove rigorously the existence (and uniqueness under certain conditions) of curved/conical traveling fronts connecting the "colder" to the "hot" states
- Derive a formula for the speed of propagation
- Use existing results on planar fronts to study asymptotic behaviour of the speed c when the amplitude of the advection is large

Curved/Conical fronts

α and β are given in $(0,\pi)$, $\alpha+\beta\le\pi$:

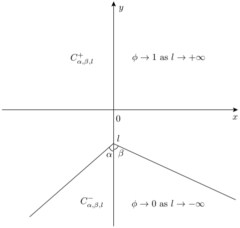


$$C^-_{\alpha,\beta,l} = \left\{(x,y)\in\mathbb{R}^2,\ \ y \le x\cot\alpha + l,\ x\le 0 \text{ and } \ y\le -x\cot\beta + l,\ x\ge 0\right\}$$

$$C^+_{\alpha,\beta,l} = \overline{\mathbb{R}^2 \setminus C^-_{\alpha,\beta,l}}$$

Curved front: definition

$$\frac{\partial u}{\partial t} = \Delta u + q(x)\frac{\partial u}{\partial y} + f(u), \text{ for all } t \in \mathbb{R},\ (x,y) \in \mathbb{R}^2, \tag{2}$$



Curved fronts traveling with a speed c

are classical solutions to (2) of the form $u(t,x,y) = \phi(x, y + ct)$

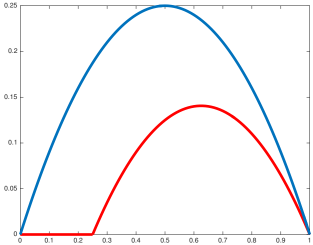
$$\lim_{l \to -\infty}\Big(\sup_{(x,y)\in C^-_{\alpha,\beta,l}} \phi(x,y)\Big) = 0, \quad \lim_{l \to +\infty}\Big(\inf_{(x,y)\in C^+_{\alpha,\beta,l}} \phi(x,y)\Big) = 1.$$

$$\Delta\phi + (q(x) - c)\partial_y\phi + f(\phi) = 0 \text{ for all } (x,y) \in \mathbb{R}^2.$$

The reaction term

Combustion type nonlinearity (red): $f(u)$ accounts for an ignition temperature θ:

$$\exists\theta \in (0,1);\ f \equiv 0 \text{ on } [0,\theta] \cup \{1\},\ f > 0 \text{ on } (\theta,1) \text{ and } f'(1) < 0. \qquad (3)$$



KPP Type nonlinearity (blue):

$$f(u) \leq f'(0)u \text{ for } 0 \leq u \leq 1 \;\&\; f \equiv 0 \text{ on } \mathbb{R} \setminus (0,1)$$

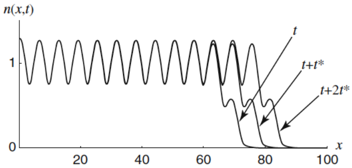
Prior works

- F. Hamel, R. Monneau, and J.-M. Roquejoffre, *École Norm. Sup. (2004)*
- H. Ninomiya and M. Taniguchi, *Disc. Cont. Dyn. Syst. A* **15** (2006)
- El Smaily, F. Hamel, and R. Huang, *Nonlinear Analysis: Theory, Methods & Applications* **74**, (2011).
- M. Taniguchi, *J. Diff. Equations (2009)*
- Wang, Zhi-Cheng; Bu, Zhen-Hui (several works in 2015, Comm. Pure Applied Analysis 2016 & a 2017 preprint)

In the above references:

- either no advection term is present in the model
- or advection is present but the nonlinearity f is not of "ignition type".
- The fact that combustion nonlinearity is not concave on the interval $[0, 1]$ is what complicates the proof of existence.

Planar fronts



Theorem (Berestycki & Hamel, *CPAM*, 2002)

$$\frac{\partial u}{\partial t} = \nabla \cdot (M \nabla u) + q(X) \sin \gamma \frac{\partial u}{\partial Y} + f(u), \quad t \in \mathbb{R}, \ (X,Y) \in \mathbb{R}^2$$
$$u(t,X,Y) \underset{Y \to -\infty}{\longrightarrow} 0, \quad u(t,X,Y) \underset{Y \to +\infty}{\longrightarrow} 1, \tag{4}$$

where $M(X,Y)$ *is uniformly elliptic,* q *and* M *are periodic in the* X*-variable, admits* planar traveling front(s) $u(t,X,Y) = \varphi(X, Y+ct)$

- in the KPP case: *A minimal speed* $c^*_{M,q\sin\gamma,f} > 0$. *PTFs exist for any* $c \geq c^*$.
- in the combustion case*:* $\exists$ Unique speed $c_\theta := c_{M,q,f} > 0$ *and a unique front* $u(t,X,Y)$ *with speed* c_θ. *This front is unique up to a shift in* t.

Connection to pulsating traveling fronts

- The reaction-diffusion equation with $u(t,x,y) = \phi(x, y+ct)$ becomes

$$\Delta\phi + (q(x) - c)\partial_y\phi + f(\phi) = 0 \ \text{ for all } \ (x,y) \in \mathbb{R}^2. \tag{5}$$

- If $\phi_1(x,y) = \varphi_\beta(x, x\cos\beta + y\sin\beta)$, $\phi_2(x,y) = \varphi_\alpha(x, x\cos\alpha + y\sin\alpha)$.

$$X = x, \quad Y = x\cos\beta + y\sin\beta, \quad Y' = -x\cos\alpha + y\sin\alpha,$$

- then the conical limiting conditions

$$\lim_{l\to-\infty}\Big(\sup_{(x,y)\in C^-_{\alpha,\beta,l}}\phi(x,y)\Big) = 0, \quad \lim_{l\to+\infty}\Big(\inf_{(x,y)\in C^+_{\alpha,\beta,l}}\phi(x,y)\Big) = 1$$

transform to conditions involving

$$\lim_{Y\to\pm\infty}\varphi_\beta(X,Y) \text{ and } \lim_{Y\to\pm\infty}\varphi_\alpha(X,Y).$$

- $$\Delta_{x,y}\phi_1 = \nabla_{X,Y}\cdot(B\nabla_{X,Y}\varphi_\beta),\ \ \Delta_{x,y}\phi_2 = \nabla_{X,Y'}\cdot(A\nabla_{X,Y'}\varphi_\alpha)$$
$$A = \begin{bmatrix} 1 & -\cos\alpha \\ -\cos\alpha & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & \cos\beta \\ \cos\beta & 1 \end{bmatrix}$$
- $$q(x)\partial_y\phi_1 = q(X)\sin\beta\,\partial_2\varphi_\beta,$$

Conical fronts: KPP case

Theorem (joint with Hamel and Huang, 2011)

Let f be a nonlinearity of KPP type*. Then, for any given α and β in $(0, \pi)$ such that $\alpha + \beta \leq \pi$, there exists a positive real number c^* such that*
i) for each $c \geq c^*$, the problem (5) admits a solution (c, ϕ);
ii) if $c < c^*$, the problem (5) *has no solution (c, ϕ).*
Moreover, the value of c^* is given by

$$c^* = \max\left(\frac{c^*_{A,q\sin\alpha,f}}{\sin\alpha}, \frac{c^*_{B,q\sin\beta,f}}{\sin\beta}\right), \tag{6}$$

where

$$A = \begin{bmatrix} 1 & -\cos\alpha \\ -\cos\alpha & 1 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 1 & \cos\beta \\ \cos\beta & 1 \end{bmatrix}. \tag{7}$$

Combustion case

Main assumptions: $q(-x) = q(x)$ and $q(x+L) = q(x)$ (for some $L > 0$).

Theorem (submitted in 2018)

Let f be a nonlinearity of combustion type. *Then, for any given α in $(0, \pi/2)$, there exists $c_\theta > 0$ and a conical traveling front $u(t,x,y) = \phi(x, y + c_\theta t)$ satisfying the conical limiting conditions* (6).
The value of c_θ is given by

$$c_\theta = \frac{c_{A, q \sin\alpha, f}}{\sin\alpha} = \frac{c_{B, q \sin\alpha, f}}{\sin\alpha}, \tag{8}$$

where

$$A = \begin{bmatrix} 1 & -\cos\alpha \\ -\cos\alpha & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{bmatrix}. \tag{9}$$

- The symmetry assumption $q(-x) = q(x)$ was not needed in the KPP case.
- The speed c_θ is unique, while a spectrum of speeds exists in the KPP case.

Sketch of the proof

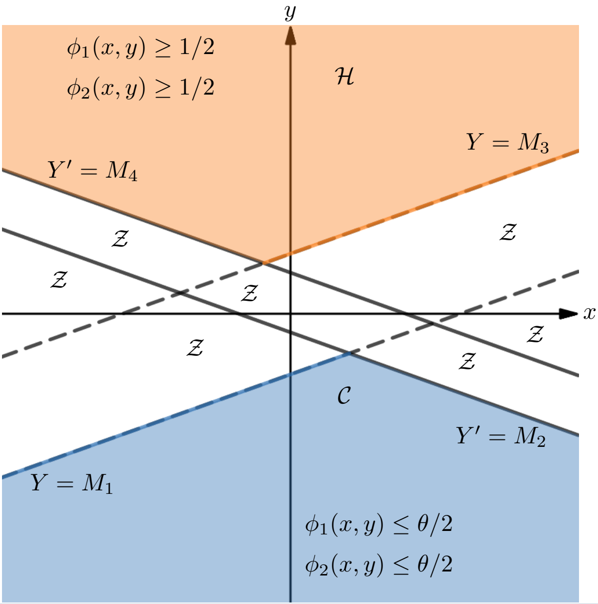
$$\phi_1(x,y) = \varphi_\beta(x, x\cos\beta + y\sin\beta),\ \phi_2(x,y) = \varphi_\alpha(x, -x\cos\alpha + y\sin\alpha).$$

- $\underline{\phi}(x,y) := \max\{\phi_1, \phi_2\}$
- For a supersolution: the complication that prevents a nominee such as $\overline{\phi}(x,y) = \phi_1 + \phi_2$ from being a supersolution is that f is not concave downwards over $[0,1]$ entirely.
- In the KPP case, where f is concave on $[0,1]$ entirely, the inequality

$$f(\phi_1 + \phi_2)$$

  helps making $\phi_1 + \phi_2$ a supersolution.
- We will see that $\overline{\phi}(x,y) = \min(H(\phi_1 + \phi_2), 1)$ for some appropriate H works as a supersolution.
- Make sure that $\overline{\phi}$ and $\underline{\phi}$ satisfy $\overline{\phi} \geq \underline{\phi}$ and also satisfy the desired limits as $l \to \pm\infty$ (conical limits)
- Use a Perron type theorem to conclude the existence of a solution trapped between $\overline{\phi}$ and $\underline{\phi}$.

The choice of H



$$\Delta\overline{\phi}(x,y)+\left(q(x)-\tfrac{c}{\sin\alpha}\right)\partial_y\overline{\phi}(x,y)+f(\overline{\phi})=$$

$$f(H(\varphi+\psi))-H'(\varphi+\psi)[f(\varphi)+f(\psi)]+$$

$$H''(\varphi+\psi)\left[(\partial_1\varphi+\partial_1\psi+\cos\alpha\partial_2\varphi+\cos\alpha\partial_2\psi)^2+\sin^2\alpha(\partial_2\varphi+\partial_2\psi)^2\right].$$

If $H''\leq 0$ in $[0,2]$, we get

$$\begin{aligned}&\Delta\overline{\phi}(x,y)+\left(q(x)-\tfrac{c}{\sin\alpha}\right)\partial_y\overline{\phi}(x,y)+f(\overline{\phi})\\ &\leq f(H(\varphi+\psi))-H'(\varphi+\psi)[f(\varphi)+f(\psi)]\\ &\quad+H''(\varphi+\psi)\left[\sin^2\alpha(\partial_2\varphi+\partial_2\psi)^2\right].\end{aligned}\tag{10}$$

Auxiliary ODE

Lemma

Let $\beta > 0$ be a positive number and let h_β (write h for simplicity) denote the unique solution to the initial value problem

$$\begin{cases} \beta H''(z) + f(H(z)) = 0 \quad \text{for } \frac{\theta}{2} < z < 2, \\ H(\theta/2) = \theta, \\ H'(\theta/2) = 2. \end{cases} \tag{11}$$

Then the following assertions hold

(a) *For any $\beta > 0$, the solution H is strictly increasing on the interval $[\theta/2, 2]$.*

(b) *For any $\beta > 0$, $H(1) > 1$.*

(c) *For any $\beta > 0$, $H(2) \geq 1$.*

Most delicate case

$(x, y) \in \mathcal{Z}$. In this region,

$$\partial_2 \phi(x, Y) \geq \mu \text{ and } \partial_2 \psi(x, Y') \geq \mu$$

whenever $(x, y) \in \mathcal{Z}$. Then, as $H' = h' > 0$, the right hand side of (10) can be bounded above as

$$f(H(\varphi + \psi)) - H'(\varphi + \psi)[f(\varphi) + f(\psi)] + H''(\varphi + \psi)\left[\sin^2 \alpha(\partial_2\varphi + \partial_2\psi)^2\right]$$
$$\leq f(H(\varphi + \psi)) + \beta H''(\varphi + \psi) \text{ provided that } \beta \leq 4\mu^2 \sin^2 \alpha$$
$$= f(h(\varphi + \psi)) + \beta h''(\varphi + \psi) \text{ provided that } \beta \leq 4\mu^2 \sin^2 \alpha$$
$$\leq 0.$$

What happens when q is large

Considering the problem

$$\Delta\phi + (Mq(x) - c)\partial_y\phi + f(\phi) = 0 \text{ for all } (x,y) \in \mathbb{R}^2, \tag{12}$$

with the limiting conditions (6), $M > 0$ is the amplitude of the advection term,
Question: Does the large amplitude flow speed-up the curved front in the case of ignition nonlinearity?
F. Hamel and A. Zlatos, Math. Ann. (2013) studied this question for pulsating traveling fronts (planar fronts).
As the speed c_θ here is expressed as $c_\theta = \frac{c_{A,q\,\mathbf{sin}\,\alpha,f}}{\sin\alpha}$, their result applies:

$$\lim_{M\to+\infty} \frac{c(\theta, M)}{M} = \lim_{M\to+\infty} \frac{c_{A,Mq\sin\alpha,f}}{M\sin\alpha}$$

is a positive number which can be computed explicitly.

What happens when q is large

This means that the speed behaves as $O(M)$ when $M \to +\infty$ and the large flow q speeds-up the fronts linearly.