Curved fronts describing flames in a shear flow

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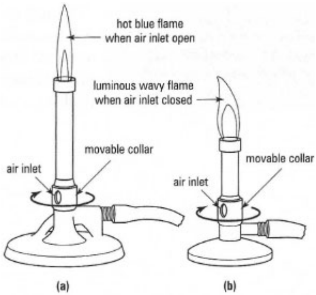
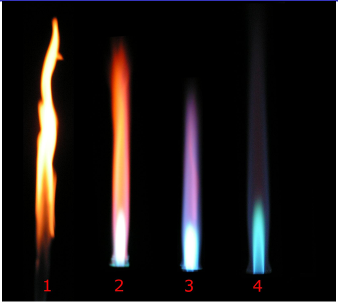
Outline

1. the problem in mathematical terms
2. Prior Works
   - Planar fronts (as part of the curved fronts construction)
3. Curved fronts in the case of combustion nonlinearity
4. planar fronts as a tool in the proof
5. Asymtotics in a shear flow with large amplitude





- A Bunsen burner, an air inlet with movable collar
- When the air inlet is open, lots of air mixes with burning fuel
- produces a "noisy, roaring" flame
- One hopes to heating things quickly by using the ADVECTION'S influence

Some Questions and goals

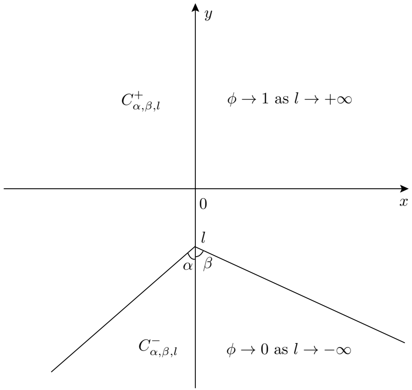
We will use the model

$$\frac{\partial u}{\partial t} = \Delta u + q(x)\frac{\partial u}{\partial y} + f(u), \text{ for all } t \in \mathbb{R},\ (x, y) \in \mathbb{R}^2, \quad (1)$$

where $u(t, x, y)$ stands for the "rescaled" temperature.

- the nonlinearity is to be discussed
- quantify the influence of the advection term "q" on the shape of the flame (the level sets of $u(t, \cdot, \cdot)$)
- prove rigorously the existence (and uniqueness under certain conditions) of curved/conical traveling fronts connecting the "colder" to the "hot" states
- derive a formula for the speed of propagation
- Use existing results on planar fronts to study asymptotic behavior of the speed c when the Amplitude of the advection is Large

curved/Conical fronts

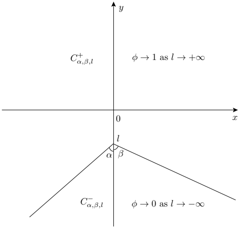


α and β are given in $(0, \pi)$, $\alpha + \beta \leq \pi$:

$$C^-_{\alpha,\beta,l} = \{(x, y) \in \mathbb{R}^2,\ \ y \leq x \cot \alpha + l,\ x \leq 0 \text{ and }\ y \leq -x \cot \beta + l,\ x \geq 0$$

$$C^+_{\alpha,\beta,l} = \overline{\mathbb{R}^2 \setminus C^-_{\alpha,\beta,l}}$$

Curved front: definition



$$\frac{\partial u}{\partial t} = \Delta u + q(x)\frac{\partial u}{\partial y} + f(u), \text{ for all } t \in \mathbb{R},\ (x,y) \in \mathbb{R}^2, \qquad (2)$$

Curved fronts traveling with a speed c

are classical solutions to (2) of the form $u(t,x,y) = \phi(x, y+ct)$ s.t.

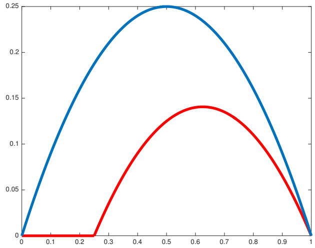
$$\lim_{l\to-\infty}\Big(\sup_{(x,y)\in C^-_{\alpha,\beta,l}} \phi(x,y)\Big) = 0, \quad \lim_{l\to+\infty}\Big(\inf_{(x,y)\in C^+_{\alpha,\beta,l}} \phi(x,y)\Big) = 1.$$

THE REACTION TERM

COMBUSTION TYPE NONLINEARITY (<u>RED</u>): $f(u)$ accounts for an IGNITION temperature θ:

$$\exists\theta \in (0,1);\ f \equiv 0 \text{ on } [0,\theta] \cup \{1\},\ f > 0 \text{ on } (\theta,1) \text{ and } f'(1) < 0. \tag{4}$$

KPP TYPE NONLINEARITY (<u>BLUE</u>):



$$f(u) \leq f'(0)u \quad \text{for } 0 \leq u \leq 1 \ \&\ f \equiv 0 \quad \text{on} \quad \mathbb{R} \setminus (0,1)$$

Several Prior works studied fronts with conical limiting conditions

Some prior works

- F. Hamel, R. Monneau, and J.-M. Roquejoffre, *École Norm. Sup. (2004)*
- H. Ninomiya and M. Taniguchi, *Disc. Cont. Dyn. Syst. A* **15** (2006)
- El Smaily, F. Hamel, and R. Huang, *Nonlinear Analysis: Theory, Methods & Applications* **74**, (2011).
- M. Taniguchi, *J. Diff. Equations (2009)*
- Wang, Zhi-Cheng; Bu, Zhen-Hui (several works in 2015, Comm. Pure Applied Analysis 2016 & a 2017 preprint)

But, in the above references:

- either no advection term is present in the model
- or advection is present but the nonlinearity f is not of "ignition type".
- technically: the fact that such nonlinearity is <u>not concave</u> on the interval $[0, 1]$ is what complicates the proof of existence.

A simple connection to planar fronts

- The reaction-diffusion equation rewritten with $u(t,x,y) = \phi(x, y+ct)$:

$$\Delta\phi + (q(x) - c)\partial_y\phi + f(\phi) = 0 \ \text{ for all } \ (x,y) \in \mathbb{R}^2. \qquad (5)$$

- Say $\phi_1(x,y) = \varphi_\beta(x, x\cos\beta + y\sin\beta)$ & $\phi_2(x,y) = \varphi_\alpha(x, x\cos\alpha + y\sin\alpha)$. Then $\Delta_{x,y}\phi_1 = \nabla_{X,Y}\cdot(B\nabla_{X,Y}\varphi_\beta)$, $\Delta_{x,y}\phi_2 = \nabla_{X,Y'}\cdot(A\nabla_{X,Y'}\varphi_\alpha)$ and $q(x)\partial_y\phi_1 = q(X)\sin\beta\,\partial_2\varphi_\beta$, $X = x$, $Y = x\cos\beta + y\sin\beta$, $Y' = -x\cos\alpha + y\sin\alpha$,
$$B = \begin{bmatrix} 1 & \cos\beta \\ \cos\beta & 1 \end{bmatrix}.$$

• The conical limiting conditions

$$\lim_{l\to-\infty}\Big(\sup_{(x,y)\in C^-_{\alpha,\beta,l}} \phi(x,y)\Big) = 0, \quad \lim_{l\to+\infty}\Big(\inf_{(x,y)\in C^+_{\alpha,\beta,l}} \phi(x,y)\Big) = 1$$

transform to conditions involving $\lim_{Y\to\pm\infty}\varphi_\beta(X,Y)$ *and* $\lim \ \varphi_\alpha(X,Y)$

Planar fronts (as an auxiliary tool)

[Berestycki & Hamel, *CPAM*, 2002; Weinberger 2002; J. Xin 2000]:

$$\frac{\partial u}{\partial t} = \mathrm{div}(M\nabla u) + q(X)\sin\gamma\frac{\partial u}{\partial Y} + f(u), \quad t\in\mathbb{R},\ (X,Y)\in\mathbb{R}^2$$

$$u(t,X,Y) \underset{Y\to-\infty}{\longrightarrow} 0, \quad u(t,X,Y) \underset{Y\to+\infty}{\longrightarrow} 1, \tag{6}$$

where $M(X,Y)$ is uniformly elliptic, q and M are periodic in the X-variable, admits *planar traveling front(s)* $u(t,X,Y) = \varphi(X, Y+ct)$

- IN THE KPP CASE: A minimal speed $c^*_{M,q\sin\gamma,f} > 0$. PTFs exist for any $c \geq c^*$. Each front (c,u) is unique up to a shift in t
- IN THE COMBUSTION CASE: There exists a UNIQUE SPEED

KPP case

Theorem (joint with Hamel and Huang, 2011)

Let f be a nonlinearity of KPP $\textsc{type}$. *Then, for any given α and β in $(0,\pi)$ such that $\alpha+\beta\le\pi$, there exists a positive real number c^* such that*
i) for each $c\ge c^*$, the problem (5)-(3) admits a solution (c,ϕ);
ii) if $c<c^*$, the problem (5)-(3) has no solution (c,ϕ).
Moreover, the value of c^* is given by

$$c^* = \max\left(\frac{c^*_{A,q\sin\alpha,f}}{\sin\alpha}, \frac{c^*_{B,q\sin\beta,f}}{\sin\beta}\right), \tag{7}$$

where

$$A = \begin{bmatrix} 1 & -\cos\alpha \\ -\cos\alpha & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & \cos\beta \\ \cos\beta & 1 \end{bmatrix}. \tag{8}$$

Existence and uniqueness

Main assumptions: $q(-x) = q(x)$ and $q(x+L) = q(x)$ (for some $L > 0$).

Theorem (preprint 2017)

Let f be a nonlinearity of COMBUSTION TYPE. *Then, for any given α in $(0, \pi/2)$, there exists $c_\theta > 0$ and a conical traveling front $u(t,x,y) = \phi(x, y + c_\theta t)$ satisfying the conical limiting conditions (3).*
The value of c_θ is given by

$$c_\theta = \frac{c_{A,q\sin\alpha,f}}{\sin\alpha} = \frac{c_{B,q\sin\alpha,f}}{\sin\alpha}, \tag{9}$$

where

$$A = \begin{bmatrix} 1 & -\cos\alpha \\ -\cos\alpha & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{bmatrix}. \tag{10}$$

Notes:

- The symmetry assumption $q(-x) = q(x)$ was not needed in the KPP case.
- The speed c_θ is unique, while a spectrum of speeds exists in the KPP case.

main steps of the proof:

- For a subsolution, take $\underline{\phi}(x, y) = \max\{\phi_1(x, y), \phi_2(x, y)\}$
- For a supersolution: take $\overline{\phi}(x, y) = \min(h(\phi_1 + \phi_2), 1)$ where (after some computations) the appropriate h turns out to be a concave down parabola with a negative curvature $-\kappa$. Part of the proof is to find a suitable range of the parameters appearing in the function h.
- Make sure that $\overline{\phi}$ and $\underline{\phi}$ satisfy $\overline{\phi} \geq \underline{\phi}$ and also satisfy the desired limits as $l \to \pm\infty$ (conical limits)
- Use a Perron type theorem to conclude the existence of a solution trapped between $\overline{\phi}$ and $\underline{\phi}$.

What happens when q is large

Considering the problem

$$\Delta\phi + (Mq(x) - c)\partial_y\phi + f(\phi) = 0 \text{ for all } (x,y) \in \mathbb{R}^2, \qquad (11)$$

with the limiting conditions (3), $M > 0$ is the amplitude of the advection term,

Question: Does the large amplitude flow speed-up the curved front in the case of ignition nonlinearity?

F. Hamel and A. Zlatos Math. Ann. (2013) studied this question for pulsating traveling fronts (planar fronts).

As the speed c_θ here is expressed as $c_\theta = \frac{c_{A,q\sin\alpha,f}}{\sin\alpha}$, their result applies:

$$\lim_{M\to+\infty}\frac{c(\theta,M)}{M} = \lim_{M\to+\infty}\frac{c_{A,Mq\sin\alpha,f}}{M\sin\alpha}$$

is a positive number which can be computed explicitly.

What happens when q is large

This means that the speed behaves as $O(M)$ when $M \to +\infty$ and the large flow q speeds-up the fronts linearly.