# Evolution in heterogeneous media with large drift

## Summer School on nonlinear evolutions

HCM, University of Bonn

Mohammad El Smaily[1]

Department of Mathematics,
American University of Beirut

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# Outline

1. Background and particular features
   - A brief description of reaction-advection-diffusion
   - Evolution in homogeneous settings
   - Heterogeneous settings (Variational formula)

2. Deterministic analysis on the influence of large advection
   - In any spatial dimension $N$
   - Sharp Criteria: speed-up by flows in 2D
   - Sharp Criteria: speed-up by flows in 3D

3. Open questions

# Introduction

- Introduced in 1937 simultaneously by Fisher, and Kolmogorov-Petrovskii-Piskunov, the celebrated **F-KPP equation** describes various reaction-diffusion phenomena which give rise to **front propagation**.
- Plays a central role in various models related to combustion, chemistry, biology and ecology
- (indeed, Fisher's original goal was to study how an advantageous allele spreads in a geographically distributed population).
- To a large extent, the F-KPP equation started the field of semi-linear parabolic differential equations
- This equation is intimately connected with a class of probabilistic models, (*branching random walks*) — in a way similar to the relation between the heat equation and the standard Brownian motion.

# How do reaction-advection-diffusion equations appear?

- Let $u(t, z)$ denote the concentration/density of certain organism or chemical at time $t$ and location $z \in \Omega \subseteq \mathbb{R}^N$ $(N = 1, 2, 3, \cdots)$
- In population dynamics, $\Omega$ is the habitat, $u(t, z)$ is always associated with a scale, like country, city, town, and street.
- In any sub-region $O \subseteq \Omega$ containing $z$,

$$\text{total population (or material)} = \int_O u(t, z) dz$$

- The Diffusion mechanism: on $\partial O$, "stuff" moves from regions with high density to others with lower density *as quickly as possible*:

$$z \in \partial O, \text{ diffusive flux vector} = J_{diffusive} = -A(z)\nabla_z u$$

- Divergence theorem:

$$\text{Total diffusive flux} = \int_{\partial O} (-A\nabla u) \cdot n \, d\sigma = + \int_O \nabla \cdot (A(z)\nabla_z u) dz$$

# Appearance of reaction and advection

- Reaction: at location $z \in O$, $u(t,z)$ can change due to factors such as birth, death, hunting, or chemical reactions. The amount of change due to such reasons is $f(t,z,u)$, the reaction term.
- So the net growth (in $O$) is $\int_O f(t,z,u(t,z))dz$
- **Some variations inside the medium result from moving flows with velocity $q$ (water, chemicals). Then,**

$$\text{Advective flux}: J_{adv} = \int_{\partial O} \vec{q}u \cdot n = -\int_O \nabla \cdot (qu)\ dz$$

- $$\frac{d}{dt}\int_O u(t,z)dz = \underbrace{\int_O \nabla \cdot (A\nabla_z u)\ dz}_{\text{diffusive flux}} \pm \underbrace{\int_O q \cdot \nabla u\ dz}_{\text{advective flux}} + \int_O f(t,z,u)dz$$
- Since $O$ is arbitrary, we can drop the integrals and get the reaction-**advection**-diffusion model

$$u_t = \nabla_z \cdot (A(z)\nabla_z u) + q \cdot \nabla u + f(t,z,u) \text{ for } (t,z) \in \mathbb{R} \times \Omega$$

# Key features

- Reaction-advection-diffusion equations enjoy a nice feature: traveling waves or pulsating traveling waves (depending on the nature of the framework).
- Pulsating traveling fronts were introduced (in the general sense) in Weinberger 2002, Berestycki-Hamel-Nadirasvilli (2002-2005)
- Before that, motivations from math-biology and ecology papers: S-K-T 1988, and J-Xin 1999.

# In purely diffusive media

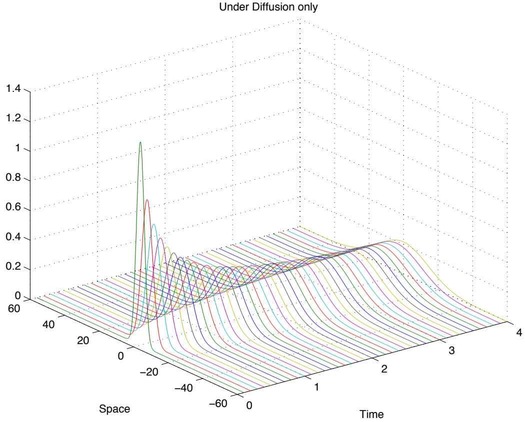
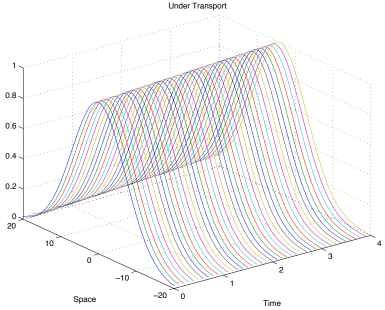


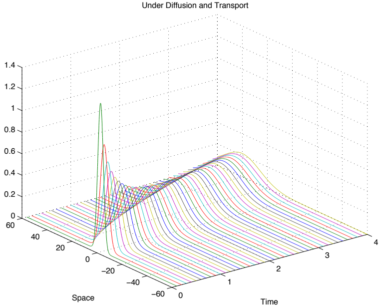

Figure: Thinking of $u_t = \mathbf{d} u_{xx}$

# Transport equation versus diffusion-advection

$u_t = \alpha u_x, \ u(0,x) = u_0(x)$

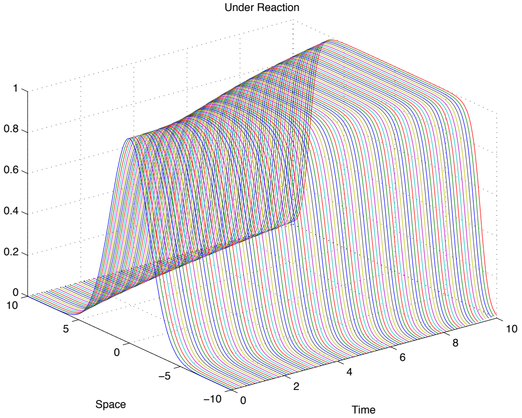



$u_t = u_{xx} + \alpha u_x$

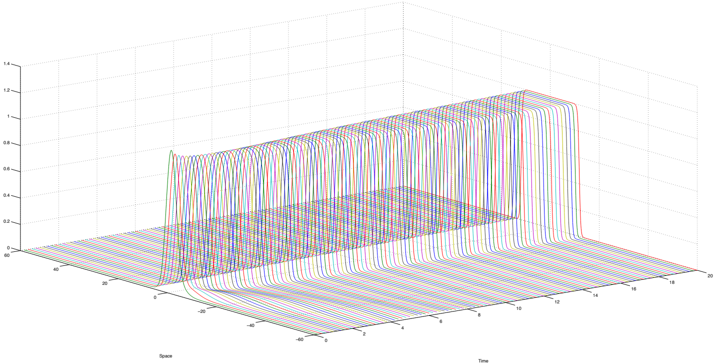


# Logistic growth

$u_t(t,x) = u(1-u)$, $u_0(x) = e^{-x^2}$. Then, $u(t,x) = \dfrac{u_0(x)e^t}{1+u_0(x)(e^t-1)}$



# Traveling fronts (space-time)



# Travelling fronts

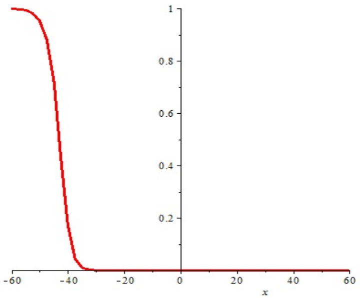
- A homogeneous reaction-diffusion equation of the form:

$$u_t(t,x) = \Delta u + f(u) \quad t \in \mathbb{R},\ x \in \mathbb{R}^N. \tag{1}$$

- $f(0) = f(1) = 0$, no advection term $(q \cdot \nabla u)$, isotropic medium.

## Definition (**Travelling Fronts**)

Given a direction $e \in \mathbb{R}^N$, *travelling fronts* propagating in the direction of $-e$ and with a speed $c \in \mathbb{R}$ were introduced as solutions of (1) in the form $u(t,x) = \phi(x \cdot e - ct) = \phi(s)$ satisfying the limiting conditions $\phi(-\infty) = 1$ and $\phi(+\infty) = 0$.



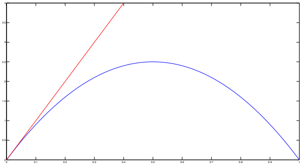
# Typical nonlinearities (KPP or logistic map)

- $f(u) = b(u) - d(u) = u - u^2$ (see $b = u$ as the birth rate, and $d = u^2$ as the death rate).
- (1) then corresponds to biological invasion of the unstable state 0 by the stable state 1 (Aronson, Weinberger, Lewis, Diekmann, SKT, many others)

$f = f(u)$ is called a homogenous KPP nonlinearity when

$$f(0) = f(1) = 0, f \equiv 0 \text{ in } \mathbb{R} \setminus [0,1],\ 0 < f(s) \leq f^{'}(0)s,\ s \in (0,1).$$

- Typical example: $f(u) = u(1-u)$ on $[0,1]$.



# First known result about travelling fronts

## Theorem (Kolmogorov-Petrovsky-Piskunov, 1937)

*Having a KPP nonlinearity, a TF exists with a speed $c$ iff $c \geq 2\sqrt{f'(0)}$. Moreover, this TF $u(t,x)$ is decreasing in $t$.*

$$c^* = 2\sqrt{f'(0)}$$

*is the minimal speed in the homogeneous case (**there is no advection.**)*

- Notice that $f'(0)$, the growth rate at $u = 0$, is the only appearance of the reaction in the formulation of the minimal speed.
- Also, $u_t = \Delta u + f(u)$ can be rewritten in this case as

$$\phi''(s) + c\phi'(s) + f(\phi(s)) = 0 \text{ for all } s \in \mathbb{R} \tag{2}$$

# A basic PDE idea proving the KPP theorem

KPP used a phase-portrait analysis and ODE tools to prove their result. A PDE approach (inspired by Nirenberg-Berestycki 1990) will make us see things more clearly (in heterogeneous settings)

- Over any interval $(-a, a) \subset \mathbb{R}$, the nondecreasing function

$$\overline{\phi}^{a,r}(s) := \min(e^{-\lambda s + r}, 1)$$

is a super-solution over $(-a, a)$ of (2) whenever $c \geq 2\sqrt{f'(0)}$ and $\lambda_1(c) \leq \lambda \leq \lambda_2(c)$, where $\lambda_1$ and $\lambda_2$ are the real solutions to

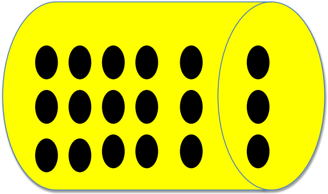
$$\lambda^2 - c\lambda + f'(0) = 0,$$

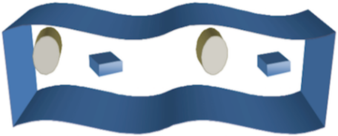
whose discriminant is equal to $c^2 - 4f'(0)$.

# In heterogeneous (periodic) media

$$\begin{cases} u_t = \nabla \cdot (A(z)\nabla u) \; + q(z) \cdot \nabla u + f(z,u), \; t \in \mathbb{R}, \; z \in \Omega, \\ \nu \cdot A(z)\nabla u = 0 \; \text{ on } \mathbb{R} \times \partial\Omega. \end{cases} \quad (3)$$

- $\Omega \subseteq \mathbb{R}^d_x \times \mathbb{R}^{N-d}_y$,
- $\Omega$ is bounded in the $y-$directions, $\Omega$ is periodic in the $x-$directions.





# Set up, Notations

- Periodicity cell $C := \Omega / L_1\mathbb{Z} \times \cdots L_d\mathbb{Z} \times \{0\}^{N-d}$
- Coefficients: $A$, $q$, and $f(x, y, u)$ are $(L_1, \cdots, L_d)-$periodic in $x$.
- The advection term $q \cdot \nabla u$: $q$ is divergence free, of zero-average $C$

## Definition (Pulsating travelling fronts, 2002)

A PTF, propagating in the direction of $-e$ with a speed $c$, is a solution

$$u(t, x, y) = \phi(s, x, y) = \phi(x \cdot e - ct, x, y)$$

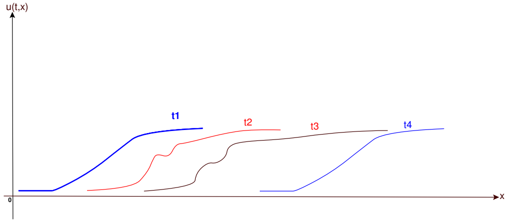
$L-$periodic in $x$, $\phi(-\infty, \cdot, \cdot) = 0$, $\phi(+\infty, \cdot, \cdot) = 1$ uniformly in $(x, y) \in \Omega$.

## Shigesada-Kawasaki-Teramoto (1988)

used these models (1D) to study the *growth and spreading* of an invading species in a forest which consists of trees planted in periodic rows.

# Pulsating Travelling Fronts

- A remark on the stationary states: All remains true when the stationary states of the equation are 0 (unstable) and $p(x, y)$ (space-dependent) as long as the domain $\Omega$ has the periodicity structure in the unbounded spatial directions



# Existence, KPP minimal speed, Variational formula

## Theorem (Berestycki and Hamel, CPAM 2002)

*For any prefixed* $e \in \mathbb{R}^d$, *there exists a minimal speed* $c^* := c^*_{\Omega,q,f}(e) > 0$ *such that a PTF with a speed* $c$ *exists if and only if* $c \geq c^*$.

$$c^*(q,e) = \min_{\lambda > 0} \frac{k(\lambda)}{\lambda}$$

$k(\lambda)$ *is the principal eigenvalue of the elliptic operator* $L_\lambda$ *defined by* $L_\lambda \psi := \Delta\psi - 2\lambda\tilde{e}\cdot\nabla\psi + q\cdot\nabla\psi + [\lambda^2 - \lambda q\cdot\tilde{e} + \zeta]\psi$ *in* $\Omega$, *on*

$$E_\lambda = \left\{\psi(x,y) \in C^2(\overline{\Omega}), \psi \text{ L-periodic in } x,\ \nu\cdot\nabla\psi = -\lambda(\nu\cdot\tilde{e})\psi \text{ on } \partial\Omega\right\}.$$

Inspired by: Aronson-Weinberger 1978, Nirenberg 1991, S-K-T 1989, , Weinberger 2000, O. Diekmann, J. Xin 2000, Weinberger-Lewis 2002, ...

# Propagation in a flow (spreading of initial-data)

A simple example: milk spreading in a cup of coffee which is being stirred.
$u(t, x, y)$ is the concentration of milk at location $(x, y)$, at time $t$.

- Which flows (stirring strategies) give an *optimal* speed up of the spreading?
- Are there flows which could block the spreading?

"Drift Paradox" [(2005) Pachepsky, Lutscher, Nisbet, Lewis MA]: Why aquatic insects faced with downstream drift are able to persist in upper **stream reaches** (1d uni-directional drift)
*"Residence in the immobile state always enhances population persistence."*

# Connecting the problem to variational analysis

Question: Study the asymptotic behaviour of the minimal speed $c^*$ when the amplitude of $q$ is large?
Known before 2012: $c^*(M) = O(M)$ as $M \to +\infty$.

## Propagation with large drift $Mq$, $(M \to +\infty)$

$$\begin{cases} u_t = \Delta u + Mq(x,y)\cdot\nabla u + f(x,y,u), \; t \in \mathbb{R}, \; (x,y) \in \Omega, \\ \nu \cdot \nabla u = 0 \; \text{ on } \mathbb{R}\times\partial\Omega. \end{cases}$$

## $k(\lambda, M)$ is the principal eigenvalue of the elliptic operator $L_\lambda$

$L_\lambda \psi := \Delta\psi - 2\lambda\tilde{e}\cdot\nabla\psi + M\,q\cdot\nabla\psi + [\lambda^2 - \lambda M\,q\cdot\tilde{e} + \zeta]\psi$ in $\Omega$,

$$c^*(M,e) = \min_{\lambda>0}\frac{k(\lambda,M)}{\lambda}$$

# Helpful test functions: first integrals

Analogous objects were introduced by Ryzhik, Novikov, Constantin and A. Zlatos (Annals of Math, 2005) "relaxation enhancing flows" in linear elliptic, parabolic problems.

## Definition (First integrals)

The family of first integrals of the advective field $q$ is defined by

$$\mathcal{I} := \quad \{w \in H^1_{loc}(\Omega),\ w \neq 0,\ w \text{ is } L-\text{periodic in } x, \text{ and } q \cdot \nabla w = 0 \text{ almost everywhere in } \Omega\}.$$

- The set $\mathcal{I}$ contains constant functions which are the "trivial first integrals". We'll see that $\mathcal{I}$ depends strongly on the space dimension $N$.
- Note that if $w \in \mathcal{I}$ is a first integral of $q$ and $\eta : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, then $\eta \circ w \in \mathcal{I}$.

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# The speed-up limit in any dimension $N$

## Theorem (El Smaily–Kirsch, Adv. in Diff. Eqs, 2012)

*Fix a unit direction* $e \in \mathbb{R}^d$. *Then,*

$$\lim_{M \to +\infty} \frac{c^*(Mq, e)}{M} = \max_{w \in \mathcal{I}} \frac{\displaystyle\int_C (q \cdot \tilde{e})\, w^2}{\displaystyle\int_C w^2}. \tag{4}$$

- This specific question had been open for more than 10 years,
- Partial answers were given by Nadirashvilli, Hamel, Berestycki (upper and lower bounds), Heinze (shear flows), Zlatoš (several works).

# Sharp criteria ($N = 2$)

- Previous theorem says that, in any dimension, the best we can expect is a linear (w.r.t amplitude $M$) speed-up
- Question: When is the limit $c^*(M)/M > 0$?
- Using probabilistic tools, Rhyzik-Novikov (2008) proved that cellular-flows cannot do a linear speed up: $q$ is cellular $\implies$ $c^*(M) = O(M^{1/4})$.

## Theorem (El Smaily - Kirsch, Adv. in Diff. Eqs)

*Assume that $N = 2$. The following two statements are equivalent:*

(i) *There exists $w \in \mathcal{I}$, such that* $\displaystyle\int_C qw^2 \neq 0.$

(ii) *There exists a periodic unbounded trajectory $T(x)$ of $q$ in $\Omega$.*

*When (ii) holds and $T_x$ is* $\mathbf{a}-$*periodic, then* $\forall\, w \in \mathcal{I},\ \displaystyle\int_C q\, w^2 \in \mathbb{R}\mathbf{a}.$

# Precise description of the "best" flows in 2D

Assume that $N = 2$. Then,

(i) If there exists no periodic unbounded trajectory of $q$ in $\Omega$, then

$$\lim_{M\to+\infty} \frac{c^*_{\Omega,M\,q,f}(e)}{M} = 0, \text{ for any unit direction } e.$$

(ii) If $\exists$ a periodic unbounded trajectory $T_x$ of $q$ in $\Omega$ then

$$\lim_{M\to+\infty} \frac{c^*_{\Omega,M\,q,f}(e)}{M} > 0 \iff \tilde{e}\cdot\mathbf{a} \neq 0.$$

# An unclear situation (yet in 2D)

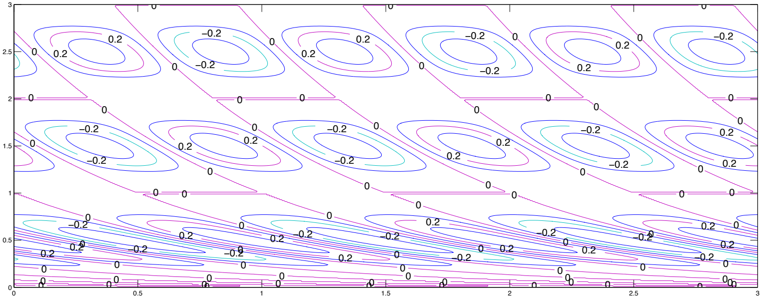
*There may exist unbounded trajectories which are not periodic, even though the vector field q is periodic*:

A periodic vector field whose unbounded trajectories are not periodic!

$$\phi(x,y) := \begin{cases} e^{-\dfrac{1}{\sin^2(\pi y)}} \sin(2\pi(x + \ln(y - [y]))) \text{ if } y \notin \mathbb{Z}, \\ 0 \text{ otherwise.} \end{cases}$$

1. $\phi$ is $C^\infty$ on $\mathbb{R}^2$, 1-periodic in $x$ and $y$. Hence, $q = \nabla^\perp \phi$ is also $C^\infty$, 1-periodic in $x$ and $y$, and $\int_{[0,1]\times[0,1]} q = 0$ with $\nabla \cdot q \equiv 0$.
2. The part of the graph of $x \mapsto e^{-x}$ lying between $y = 0$ and $y = 1$ is a trajectory of $q$, and is unbounded and not periodic.

# Shear-cellular flow?



- Our result, applied to this type of flows, yields $\lim_{M\to\infty} \frac{c^*(Mq)}{M} = 0$.
- Other results (Ryzhik et al., 2008) do not apply to this flow, as it is not cellular.
- Open: for such flows, find the sharp asymptotic regime of the speed.

# Some details about the case $N = 2$

## Proposition (Representation via a Stream function:)

*Let $d = 1$ or $2$ where $d$ is as defined before. Let $q = q(x, y) \in C^{1,\delta}(\overline{\Omega})$, L-periodic with respect to $x$ and satisfying the conditions*

$$\int_C q_i = 0, (1 \leq i \leq d) \quad \nabla \cdot q = 0 \text{ in } \Omega, \quad q \cdot \nu = 0 \text{ on } \partial\Omega.$$

*Then, there exists $\phi \in C^{2,\delta}(\overline{\Omega})$, L-periodic in $x$, such that $q = \nabla^{\perp}\phi$ in $\Omega$. Moreover, $\phi$ is constant on every connected component of $\partial\Omega$.*

- We extend $\hat{q}$ defined over $\hat{\Omega}$ to $\overline{\hat{q}}$ over $T$ by $\overline{\hat{q}} = 0$ over $T \setminus \hat{\Omega}$. In the sense of distributions $\overline{\hat{q}}$ will be also "incompressible" (thanks to the assumption $q \cdot \nu = 0$ over $\partial\Omega$).

# "Regular enough" first integrals are more available now

## Corollary

*Let* $\mathcal{J} := \{\eta \circ \phi,$ *such that* $\eta : \mathbb{R} \to \mathbb{R}$ *is Lipschitz*$\}$, $\phi$ *is s.t.* $q = \nabla^{\perp}\phi := R_{\pi/2}(\nabla\phi)$. *Then,* $\mathcal{J} \subset \mathcal{I}$.

- Useful notation:

$$T := \mathbb{R}^2/(L_1\mathbb{Z} \times L_2\mathbb{Z}) \text{ and } \hat{\Omega} := \Omega/(L_1\mathbb{Z} \times L_2\mathbb{Z}) \text{ if } d = 2,$$

$$T := \mathbb{R}^2/\left(L_1\mathbb{Z} \times \{0\}\right) \text{ and } \hat{\Omega} := \Omega/\left(L_1\mathbb{Z} \times \{0\}\right) \text{ if } d = 1.$$

- $\nabla^{\perp}\phi \cdot \nu = q \cdot \nu = 0$ on $\partial\Omega \Rightarrow \phi$ is constant on every connected component of $\partial\Omega$.
- $\tilde{\phi}$ is constant on every connected component of $T \backslash \hat{\Omega}$.

# The first integrals of the form $w = \eta \circ \phi$

- $\forall\, w \in \mathcal{J},$ we have $\displaystyle\int_C (q \cdot \tilde{e}) w^2 = 0:$
- Note that

$$w = \eta \circ \phi \text{ and } q = \nabla^{\perp}\phi \implies \int_C (q \cdot \tilde{e}) w^2 = \tilde{e} \cdot \int_C \left(\nabla^{\perp}\phi\right) \eta^2(\phi)$$
$$= \tilde{e} \cdot R \int_C \nabla\left(F \circ \phi\right) = \tilde{e} \cdot R \int_{\hat{\Omega}} \nabla(F \circ \tilde{\phi}).$$

where $R$ the matrix the rotation of angle $\pi/2$, $F' = \eta^2$,

- $\tilde{\phi}$ is constant on every connected component of $T \backslash \hat{\Omega}$, and so is $F \circ \tilde{\phi}$.

$$\text{Hence, } \int_{T \backslash \hat{\Omega}} \nabla\left(F \circ \tilde{\phi}\right) = 0.$$

- $\displaystyle\int_C (q \cdot \tilde{e}) w^2 = \tilde{e} \cdot R \int_T \nabla\left(F \circ \tilde{\phi}\right) = 0$ as $T$ has no boundary.

# Sketch of the proof of the sharp criterion in 2d

## Lemma (El Smaily-Kirsch)

*Let $\hat{V}$ be an open subset of $\hat{\Omega}$, and $\hat{\phi}$ given by $\hat{q} = \nabla^{\perp}\hat{\phi}$. Suppose that:*

- $\hat{q}(\hat{x}) \neq 0$ *for all* $\hat{x} \in \hat{V}$,
- *The level sets of* $\hat{\phi}$ *in* $\hat{V}$ *are all connected.*

*Then, for every $w \in \mathcal{I}$, there exists a continuous function $\eta : \hat{\phi}(\hat{V}) \to \mathbb{R}$ such that* $\boxed{\hat{w} = \eta \circ \hat{\phi}}$ *on* $\hat{V}$.

- Define the set of "regular trajectories" in $\hat{\Omega}$:

$$\hat{U} := \left\{ \hat{x} \in \hat{\Omega} \text{ such that } T(\hat{x}) \text{ is well defined and closed in } \overline{\hat{\Omega}} \right\}.$$

## Lemma

*The set $\hat{U}$ is exactly the union of the trajectories that are simple closed curves in $\hat{\Omega}$.*

# The bad set is not bad (Sard's lemma & Co-area formula)

$$W := \left\{ \hat{x} \in \hat{\Omega} \text{ such that } \hat{\phi}(\hat{x}) \text{ is a critical value of } \hat{\phi} \right\}.$$

- $\displaystyle\int_C qw^2 = R\int_C (\nabla\phi)w^2 = R\int_{\hat{\Omega}} (\nabla\hat{\phi})\hat{w}^2.$
- Co-area $\Longrightarrow \left|\displaystyle\int_W \hat{w}^2\nabla\hat{\phi}\right| \leq \int_W \hat{w}^2|\nabla\hat{\phi}| = \int_{\hat{\phi}(W)}\left(\int_{\hat{\phi}^{-1}(t)} \hat{w}^2(x)\right)dt.$
- From Sard's Lemma, since $\hat{\phi}$ is $C^2$, $\mathcal{L}^1(\hat{\phi}(W)) = 0$ ($\mathcal{L}^1$ denotes the Lebesgue measure on $\mathbb{R}$.)
- We then get

$$\int_W \hat{w}^2\nabla\hat{\phi} = 0.$$

• Since $\hat{\Omega}\backslash W \subset \hat{U} \subset \hat{\Omega}$, we get (letting $\hat{U}_i$ be the connected compts of $\hat{U}$)

$$\int_C qw^2 = R\left(\int_{\hat{\Omega}}(\nabla\hat{\phi})\hat{w}^2\right) = R\left(\int_{\hat{U}}(\nabla\hat{\phi})\hat{w}^2\right) = R\left(\sum_i \int_{\hat{U}_i}(\nabla\hat{\phi})\hat{w}^2\right).$$

## Lemma (El Smaily-Kirsch)

*Let $\hat{U}_i$ as in the previous definition. Then,*
*(i) all the level sets of $\hat{\phi}$ in $\hat{U}_i$ are connected,*
*(ii) $\partial\hat{U}_i$ has exactly two connected components $\hat{\gamma}_1$ and $\hat{\gamma}_2$ such that*

$$\hat{\phi}(\hat{\gamma}_1) = \sup_{\hat{x}\in\hat{U}_i} \hat{\phi}(\hat{x}) \text{ and } \hat{\phi}(\hat{\gamma}_2) = \inf_{\hat{x}\in\hat{U}_i} \hat{\phi}(\hat{x}).$$

The previous Lemmas lead to $\eta_i$ continuous such that

$$\int_{\hat{U}_i} (\nabla\hat{\phi})\hat{w}^2 = \int_{\hat{U}_i} (\nabla\hat{\phi})\eta_i^2(\hat{\phi}).$$

We define the function $F_i$ by $F_i' = \eta_i^2$ and $F_i(0) = 0$, and we obtain

$$\int_{\hat{U}_i} (\nabla\hat{\phi})\hat{w}^2 = \int_{\hat{U}_i} \nabla F_i(\hat{\phi}).$$

## Due to the condition $q \cdot \nu = 0$ on $\partial\Omega$,

trajectories of $q$ follow the boundary, and this led us to: $\gamma_1$ (resp. $\gamma_2$) is either a connected component of $\partial\hat{\Omega}$ or contains a critical point of $\hat{\phi}$.

# Cases $N \geq 3$, incompressibile flows are measure preserving

$$\begin{cases} \dfrac{d}{ds}\Phi(s,x) = & q(\Phi(s,x)), \\ \Phi(0,x) = & \Phi_0(x) = x. \end{cases}$$

- $N \geq 3$ allows incompressible flows to have more degrees of freedom and this leads to more randomness in the structure of their streamlines
- The above makes it complicated to study the level sets (surfaces) which are tangent to these trajectories.
- Since $q$ is incompressible, the Jacobian $J(s,x) := \det[\nabla_x \Phi(s,x)]$ is equal to 1 (at any $s$ and any $x \in \Omega$).
- Relevant work: "Relaxation enhancing flows" by Constantin-Novikov-Ryzhik-Zlatoš (Annals of Math. 2005)

## Definition (Ergodic components of a vector field, $N \geq 3$)

Assume $N \geq 3$. A set $V \subseteq \Omega$ is called an ergodic component of the vector field $q$ if $V$ is Lebesgue measurable with $\mathcal{L}^N(V) > 0$, $V$ is stable by the flow of $q$ and

$$(W \subset V \text{ and } W \text{ stable by the flow of } q) \Rightarrow (\mathcal{L}^N(W) = 0 \text{ or } \mathcal{L}^N(V \backslash W) = 0.$$

An ergodic component in $\Omega$ produced by the advection $q$ is, in a sense, *minimal*, up to a set of measure zero, in the family of sets which are *stable* by the flow associated to $q$.

## Remark ($N = 2$)

*An Incompressible field q will have no ergodic components in 2D.*

# First integrals and ergodic components ($N \geq 3$)

## Lemma

*Assume that $\Omega \subseteq \mathbb{R}^N$ is an open connected domain. Let $w$ be a first integral of $q$ on $\Omega$ and $I$ a measurable subset of $\mathbb{R}$. Then, up to a set of measure 0, $w^{-1}(I)$ is stable by the flow of $q$. Furthermore,*

$$\forall t \in \mathbb{R}, \quad \mathcal{L}^N\left(\Phi_t(w^{-1}(I))\Delta\left(w^{-1}(I)\right)\right) = 0,$$

*where $\Delta$ stands for the symmetric difference and $\Phi$ is the flow associated to $q$.*

## Theorem (El Smaily-Kirsch, 2014)

*Let $\Omega$ be an open subset of $\mathbb{R}^N$ (or more generally an $N$-dimensional manifold, like a flat torus). Let $q \in C^{1,\delta}(\overline{\Omega})$ be a divergence-free vector field, and $w$ be a first integral of $q$. Then, $w$ is constant a.e. on any ergodic component of the flow.*

# Existence of zero average flows with ergodic components

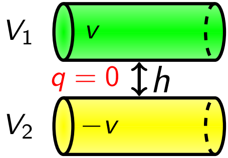



Figure: $N = 3$, two cylindrical ergodic components of $q$ aligned in the same direction with a gap of height $h \geq 0$ in between.

- Arnold-Beltrami-Childress flows (ABC flows)
- Results of Hu, Pesin and Talatiskaya (2004)
- (Katok 1979, *Annals of Math*)

confirm the existence of incompressible flows $v$ which satisfy the assumptions of our theorem.

# Criterion for linear speed-up ($N = 3, 4$)

## Theorem (El Smaily-Kirsch, Arch. for Rational Mech. & Anal., 2014)

- $\Omega = \mathbb{R}^N$ *or* $\mathbb{R} \times \omega$ *where* $N \in \{3, 4\}$, $\omega \subseteq \mathbb{R}^{N-1}$ $V_1 := \mathbb{R} \times D_1$ *and* $V_2 := \mathbb{R} \times D_2$ *be subsets of* $\Omega$ *in the same direction* $e = (1, 0, \cdots, 0)$.
- *Let* $q$ *be a N-dimensional incompressible flow:* $q|_{V_1} = v$, $q|_{V_2} = -v$, $q \equiv 0$ *on* $\mathbb{R}^N \setminus (\overline{V_1 \cup V_2})$, *(q will have zero-average over C)*
- *if q admits two ergodic components* $V_1$ *and* $V_2$ *then*

$$0 < \lim_{M \to +\infty} \frac{c^*_{\Omega,A,Mq,f}(e)}{M} < \infty \text{ if and only if } h := dist(\overline{V_1}, \overline{V_2}) > 0. \quad (5)$$

*In particular, if the ergodic components,* $V_1$ *and* $V_2$ *are tangent to each other, we have* $\lim_{M \to +\infty} \frac{c^*_{\Omega,A,Mq,f}(e)}{M} = 0$.

# Perspectives

- The large drift problem: full classification of “optimal incompressible flows” in dimension $N = 3$ (on going).
- Random media: via deterministic approaches (help from probability and stochastic analysis, works of Martin Hairer).

# Thank you for your attention