Submission of your homework assignment will be via Blackboard. Please scan all your work into one single PDF file with multiple pages and upload the file on Blackboard (learn.unbc.ca). Late assignments will be stamped "late" by the system. Early submission of your homework is possible/accepted (online). PDF is the only accepted format. A smartphone can do high quality scans of your written solutions. There is no need to use a photocopy service to scan your homework. The important is that your writing is readable.

Problem 1.

Consider the integral equation

$$f(x) = 1 + \frac{1}{\pi}\int_{-a}^{a} \frac{1}{1+(x-y)^2} f(y)\,dy \quad \text{for } -a \le x \le a, \tag{1}$$

where the functions f in the above equation are $f : [-a, a] \to \mathbb{R}$.

(a) Prove that (1) has a unique bounded, continuous solution for every $0 < a < \infty$.

(b) Prove that the solution obtained in Part (a) is nonnegative.

(c) Comment on the case where $a = \infty$.

Problem 2.

Prove that there is a unique solution to the following nonlinear BVP when the constant λ is sufficiently small,

$$\begin{cases} -u'' + \lambda \sin u = f(x), \\ u(0) = 0, \quad u(1) = 0. \end{cases} \tag{2}$$

In the above problem, $f : [0, 1] \to \mathbb{R}$ is a given continuous function. Write out the first few iterates of a uniformly convergent sequence of approximations, beginning with $u_0 = 0$.
Help. Reformulate the problem as a nonlinear integral equation.