Submission of your homework assignment will be via Blackboard. Please scan all your work into one single PDF file with multiple pages and upload the file on Blackboard (learn.unbc.ca). Late assignments will be stamped "late" by the system. Early submission of your homework is possible/accepted (online). PDF is the only accepted format. A smartphone can do high quality scans of your written solutions. There is no need to use a photocopy service to scan your homework. The important is that your writing is readable.

Problem 1.

Let $X = C([0,1])$. Let $\{f_n\}_n$ be the sequence of functions defined by:
$f_n(x) = 0$ for $0 \leq x \leq \dfrac{(1-1/n)}{2}$; $f_n(x) = 1$ for $x \geq 1/2$; $f_n(x) = 2n(x-1/2)+1$ for $\dfrac{(1-1/n)}{2} \leq x \leq 1/2$.
We know that $\|\cdot\|_2$ defined by

$$\|f\|_2 := \left(\int_0^1 (f(x))^2\, dx\right)^{1/2}$$

is a norm on X. Recall the definition of $\|f\|_\infty$ for $f \in X$. Also, remember that $(X, \|\cdot\|_\infty)$ is a Banach space (hence a complete normed vector space).
In this problem, we will see if completeness remains to be the case once we change from $\|\cdot\|_\infty$ to the norm $\|\cdot\|_2$ on X.

1. Show that $\{f_n\}_n$ is a Cauchy sequence in X with respect to $\|\cdot\|_2$.
2. Show that $\{f_n\}_n$ converges to a function f in the norm $\|\cdot\|_2$ and determine that function f.
3. Does f belong to X?
4. Is $(X, \|\cdot\|_2)$ a complete normed vector space? Justify.